EXTRACTION PROBLEM OF THE PELL SEQUENCE

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1. INTRODUCTION

Let A be an alphabet and let A^* be the free monoid over A. Let $A^+ = A^* \setminus \{\varepsilon\}$, where ε denotes the empty word. For $w \in A^*$, let $|w|$ denote the length of w. Let $|\varepsilon| = 0$. A word x is said to be a *prefix* of a finite or infinite word w over A if $x \in A^+$ and there is a word y such that $w = xy$. The finite or infinite word y is called a *suffix* of w. Let R be the *reversion operator* on A^+ defined by $R(c_1c_2\ldots c_n) = c_n\ldots c_2c_1$, where $c_i \in A$, $1 \le i \le n$, $n \ge 1$.

Let α be an irrational number between 0 and 1. The *characteristic sequence* (or word) of α is an infinite binary sequence f whose n^{th} term is $[(n+1)\alpha] - [n\alpha]$, $n \ge 1$. It will be regarded as an infinite word over the alphabet $\{0, 1\}$. Let s_m denote the prefix of f of length m and let f_m denote the suffix of f with $f = s_m f_m$, $m > 0$. Let $f_0 = f$. The characteristic sequence of $(\sqrt{5}-1)/2$ (resp., $\sqrt{2}-1$) is called the *golden sequence* (resp., *Pell sequence*).

Hofstadter [9] introduced the concept of aligning two words u and v over A (see also [3], [8]). The idea is to try to match each term (letter) in v with a term in u. After a term in v has been matched with a term in u, one looks for the earliest match to the next term in v. Those terms in u that are skipped over form the extracted word $\langle u, v\rangle$. The following example illustrates this concept.

$$
\begin{array}{rccccccccccc}
u: & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
v: & & 1 & 1 & & 0 & & 0 & & 1 & & 0 \\
\langle u, v\rangle: & 0 & & & 1 & & 1 & & 0 & & 1 &
\end{array}
$$

Originally, Hofstadter considered the problem of aligning f_m with f, where f is the characteristic sequence of an irrational number α. He conjectured that $\langle f_m, f\rangle = f_{m-2}$, $m \ge 2$. For $\alpha = (\sqrt{5}-1)/2$, Hendel and Monteferrante [8] solved this problem completely. They determined the set M of all integers $m \ge 2$ for which $\langle f_m, f\rangle = f_{m-2}$ and they proved that, if $m \ge 2$ and $m \notin M$, then $\langle f_m, f\rangle = 0f_{m-1}$. For example, $\langle f_5, f\rangle = f_3$ and $\langle f_4, f\rangle = 0f_3 \ne f_2$. The extraction problems $\langle f, f_m\rangle$ and $\langle f_m, f_n\rangle$ were first considered by Chuan [3] who proved that $\langle f, f_m\rangle = R(s_m)$, $m \ge 1$, and that $\langle f_m, f_n\rangle$ differs either from $\langle f_{m-n}, f\rangle$ (if $m > n$) or from $\langle f, f_{n-m}\rangle$ (if $m < n$) by at most the first letter. Using a concatenation lemma (Lemma 3 of [8]) and some representation theorems (Section IV of [7]), Hendel [7] also formulated and proved an extraction conjecture for $\langle f_m, f\rangle$ and $\langle f, f_m\rangle$ when $\alpha = \sqrt{2}-1$, for an infinite set of m.

In this paper, we shall use a special case of a powerful representation theorem that Chuan discovered recently [5] to prove that the following conjecture is true when $\alpha = \sqrt{2}-1$.

Conjecture: Let α be an irrational number between 0 and 1 and let f be its characteristic sequence. Then $\langle f, f_m\rangle = R(s_m)$, $m \ge 1$.

It follows from the representation lemmas in Section 2 that this conjecture has an equivalent formulation described below. Let $[0, a_1+1, a_2, \ldots]$ be the continued fraction expansion of α. Define the sequence $\{u_n\}$ of words over the alphabet $\{0, 1\}$ by

$$u_0 = 0,\ u_1 = 10^{a_1},\ u_n = u_{n-2}u_{n-1}^{a_n} \quad (n \geq 2).$$

Equivalent Formulation (Subtraction Rule of Exponents): If $n \geq 1$, $r_1, r_2, \ldots$ is an infinite sequence of integers with $0 \leq r_i \leq a_i$ $(i \geq 1)$ and $r_i = 0$ $(i > n)$, then

$$\left\langle u_0u_1^2u_2^2 \ldots,\ u_0^{1-r_1}u_1^{2-r_2}u_2^{2-r_3} \ldots \right\rangle = u_0^{r_1}u_1^{r_2} \ldots u_{n-1}^{r_n}.$$

2. PRELIMINARIES

Let $u = a_1a_2 \ldots a_n$, $v = b_1b_2 \ldots b_m$, and $e = c_1c_2 \ldots c_p$, where a_i, b_j, $c_k \in A$, n, $m > 0$, $p \geq 0$, and $n = m + p$. As in [8], we say that *u aligns with v with extraction e* if there are integers $j_1, j_2, \ldots, j_p$ such that

$$u = (b_1 \ldots b_{j_1})c_1(b_{j_1+1} \ldots b_{j_2})c_2 \ldots c_p(b_{j_p+1} \ldots b_m),$$

where $0 \leq j_1 \leq j_2 \leq \cdots \leq j_p < m$ and $c_i \neq b_{j_i+1}$ for $1 \leq i \leq p$. Here $b_1 \ldots b_k = \varepsilon$ if $k < i$. This relationship is called an *alignment* and is denoted by $\langle u, v\rangle = e$. Clearly, we have $\langle u, u\rangle = \varepsilon$.

Let u, v, and e be (possibly infinite) words over A. If $\{u_n\}$, $\{v_n\}$, and $\{e_n\}$ are sequences of finite words such that $\langle u_n, v_n\rangle = e_n$, $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty} v_n = v$, and $\lim_{n\to\infty} e_n = e$, we say that *u aligns with v with extraction e*. This alignment is also denoted by $\langle u, v\rangle = e$.

The goal of this paper is to prove the following theorem.

Theorem 2.1: ***(a)*** Let $\alpha = \sqrt{2} - 1$ and let *f* be the characteristic sequence of α. Then $\langle f, f_m\rangle = R(s_m)$ for all $m \geq 1$.

(b) (Subtraction rule of exponents) If $n \geq 1$, $r_1, r_2, \ldots$ is an infinite sequence of integers with $0 \leq r_1 \leq 1$, $0 \leq r_i \leq 2$ $(2 \leq i \leq n)$, and $r_i = 0$ $(i > n)$, then

$$\left\langle u_0u_1^2u_2^2 \ldots,\ u_0^{1-r_1}u_1^{2-r_2}u_2^{2-r_3} \cdots \right\rangle = u_0^{r_1}u_1^{r_2} \ldots u_{n-1}^{r_n}.$$

To prove this theorem, we need the following concatenation lemma and three basic representation lemmas (Lemmas 2.3-2.5).

Lemma 2.2 (see [8]): If $p > 1$, u_n, v_n, $e_n \in A^+$ and $\langle u_n, v_n\rangle = e_n$, $1 \leq n \leq p$, then

$$\left\langle \prod_{n=1}^{p} u_n,\ \prod_{n=1}^{p} v_n \right\rangle = \prod_{n=1}^{p} e_n.$$

Here $\prod_{n=1}^{p} x_n$ denotes $x_1x_2 \ldots x_p$, where $x_1, x_2, \ldots, x_p \in A^+$. The result also holds if u_p and v_p are infinite words.

Throughout the rest of this section, let α be an irrational number between 0 and 1 with continued fraction $\alpha = [0, a_1+1, a_2, \ldots]$ and let *f* be its characteristic sequence. Let

$$\begin{aligned} q_0 &= 1, & q_1 &= a_1+1, & q_n &= a_nq_{n-1} + q_{n-2}, \\ x_0 &= 0, & x_1 &= 0^{a_1}1, & x_n &= x_{n-1}^{a_n}x_{n-2}, \\ u_0 &= 0, & u_1 &= 10^{a_1}, & u_n &= u_{n-2}u_{n-1}^{a_n},\ n \geq 2. \end{aligned}$$

Note that $\{q_n\}$ is a sequence of positive integers and $\{x_n\}$ and $\{u_n\}$ are sequences of α-words over the alphabet $\{0, 1\}$ (see [4] for a definition of α-word) and $u_n = R(x_n)$, $n \geq 0$.

Lemma 2.3 (see [6]): Every positive integer m can be expressed uniquely as $m = \sum_{i=1}^{n} r_i q_{i-1}$, where $0 \leq r_i \leq a_i$ $(1 \leq i \leq n)$, $r_n \neq 0$, and $r_{i-1} = 0$ whenever $r_i = a_i$ $(2 \leq i \leq n)$.

The expression of m in Lemma 2.3 is called the *generalized Zeckendorf representation* of m in the q_i's. When $\alpha = (\sqrt{5}-1)/2 = [0, 1, 1, \ldots]$, it is the *Zeckendorf representation* and $q_i = F_{i+1}$. When $\alpha = \sqrt{2}-1 = [0, 2, 2, \ldots]$, it is also called the *Pellian representation* of m in the Pell numbers [2, 10, 11]. If $m = \sum_{i=1}^{n} r_i q_{i-1}$, where $0 \leq r_i \leq a_i$ $(1 \leq i \leq n)$, the sequence $r_1 r_2 \ldots r_n$ is called a *code* of m with respect to α (or the q_i's).

A representation of prefixes s_m of f in terms of the x_i's is given in the following lemma.

Lemma 2.4 (see [5]): Let $m = \sum_{i=1}^{n} r_i q_{i-1}$, where $0 \leq r_i \leq a_i$ $(1 \leq i \leq n)$. Then

$$s_m = x_{n-1}^{r_n} \ldots x_1^{r_2} x_0^{r_1} = R(u_0^{r_1} u_1^{r_2} \ldots u_{n-1}^{r_n}).$$

We remark that a special case of Lemma 2.4 in which the representation of m is the generalized Zeckendorf representation has been obtained by Brown [1].

In the following lemma, f and its suffixes f_m are expressed in terms of the u_n's.

Lemma 2.5 (see [5]): Let $m = \sum_{i=1}^{\infty} r_i q_{i-1}$, where $0 \leq r_i \leq a_i$ $(i \geq 1)$. Then

$$f = u_0^{a_1} u_1^{a_2} u_2^{a_3} \ldots,$$
$$f_m = u_0^{a_1-r_1} u_1^{a_2-r_2} u_2^{a_3-r_3} \ldots.$$

Note that when $\alpha = (\sqrt{5}-1)/2$, the representations of f and f_m given here reduce to the ones used in [3] and [8].

3. PROOF OF THEOREM 2.1

In this section we restrict our attention to the irrational number $\alpha = \sqrt{2}-1 = [0, 2, 2, \ldots]$. The sequences $\{q_n\}$, $\{x_n\}$, and $\{u_n\}$ defined in Section 2 now become

$$\begin{aligned} &q_0 = 1, \quad q_1 = 2, \quad q_n = 2q_{n-1} + q_{n-2}, \\ &x_0 = 0, \quad x_1 = 01, \quad x_n = x_{n-1}^2 x_{n-2}, \\ &u_0 = 0, \quad u_1 = 10, \quad u_n = u_{n-2} u_{n-1}^2, \; n \geq 2. \end{aligned} \tag{1}$$

We first prove some alignments that involve the u_n's.

Lemma 3.1:

(a) $\langle u, u \rangle = \varepsilon$ for all finite or infinite word u over $\{0, 1\}$.
(b) $\langle u_{n-1} u_n, u_n \rangle = u_{n-1}$ $(n \geq 1)$.
(c) $\langle u_n^2, u_{n-1} u_n \rangle = u_{n-2} u_{n-1}$ $(n \geq 2)$.
(d) $\langle u_{n-1}^2 u_n^2, u_n^2 \rangle = u_{n-1}^2$ $(n \geq 2)$.
(e) $\langle u_0 u_1^2 \ldots u_n^2, u_1 \ldots u_{n-1} u_n^2 \rangle = u_0 u_1 \ldots u_{n-1}$ $(n \geq 1)$.
(f) $\langle u_n^2 u_{n+1}^2 \ldots u_{n+p}^2, u_n \ldots u_{n+p-1} u_{n+p}^2 \rangle = u_n u_{n+1} \ldots u_{n+p-1}$ $(n \geq 1, p \geq 1)$.
(g) $\langle u_n^2 u_{n+1}^2 \ldots u_{n+p}^2, u_{n+1} \ldots u_{n+p-1} u_{n+p}^2 \rangle = u_n^2 u_{n+1} \ldots u_{n+p-1}$ $(n \geq 1, p \geq 2)$.

Proof:

(a) By definition.

(b)-(d) Clearly, the results hold for $n \le 3$. Let $k \ge 3$. Suppose that (b)-(d) hold for all $n \le k$. Then:

(i) $\langle u_k u_{k+1}, u_{k+1}\rangle$

$= \langle u_{k-2}u_{k-1}, u_{k-1}\rangle\langle u_{k-1}u_{k-1}u_k u_k, u_k u_k\rangle$ [by (1) and Lemma 2.2]

$= u_{k-2}u_{k-1}u_{k-1}$ [by the inductive hypothesis of (b) and (d)]

$= u_k$.

(ii) $\langle u_{k+1}u_{k+1}, u_k u_{k+1}\rangle$

$= \langle u_{k-1}u_k, u_k\rangle\langle u_k u_{k+1}, u_{k+1}\rangle$ [by (1) and Lemma 2.2]

$= u_{k-1}u_k$ [by (i) and the inductive hypothesis of (b)].

(iii) $\langle u_k^2 u_{k+1}^2, u_{k+1}^2\rangle$

$= \langle u_{k-2}u_{k-1}, u_{k-1}\rangle\langle u_{k-1}u_k, u_k\rangle\langle u_{k+1}^2, u_k u_{k+1}\rangle$ [by (1) and Lemma 2.2]

$= u_{k-2}u_{k-1}u_{k-1}u_k$ [by the inductive hypothesis of (b) and (ii)]

$= u_k^2$.

Therefore, (b)-(d) hold.

(e) $\langle u_0 u_1^2 \dots u_n^2, u_1 u_2 \dots u_{n-1}u_n^2\rangle$

$= \langle u_0 u_1, u_1\rangle\langle u_1 u_2, u_2\rangle\cdots\langle u_{n-1}u_n, u_n\rangle\langle u_n, u_n\rangle$ [by (1) and Lemma 2.2]

$= u_0 u_1 \dots u_{n-1}$ [by (b) and (a)].

(f) $\langle u_n^2 u_{n+1}^2 \dots u_{n+p}^2, u_n u_{n+1} \dots u_{n+p-1}u_{n+p}^2\rangle$

$= \langle u_n, u_n\rangle\langle u_n u_{n+1}, u_{n+1}\rangle\langle u_{n+1}u_{n+2}, u_{n+2}\rangle$

$\cdots\langle u_{n+p-1}u_{n+p}, u_{n+p}\rangle\langle u_{n+p}, u_{n+p}\rangle$ [by (1) and Lemma 2.2]

$= u_n u_{n+1} \dots u_{n+p-1}$ [by (a) and (b)].

(g) $\langle u_n^2 u_{n+1}^2 \dots u_{n+p}^2, u_{n+1} \dots u_{n+p-1}u_{n+p}^2\rangle$

$= \langle u_n^2, u_{n-1}u_n\rangle\left(\prod_{i=n}^{n+p-2}(\langle u_{i-1}u_i, u_i\rangle\langle u_i u_{i+1}, u_i u_{i+1}\rangle)\right)\langle u_{n+p}^2, u_{n+p-1}u_{n+p}\rangle$ [by (1) and Lemma 2.2]

$= x\left(\prod_{i=n}^{n+p-2} u_{i-1}\right)u_{n+p-2}u_{n+p-1}$ [by (a), (b), and (c)]

$= u_n^2 u_{n+1} \dots u_{n+p-1}$.

Here

$$x = \begin{cases} 1 & \text{if } n = 1, \\ u_{n-2}u_{n-1} & \text{if } n > 1. \end{cases}$$

Lemma 3.2: Let $n \ge 1$. Let $0 \le r_1 \le 1$, $0 \le r_i \le 2$ $(2 \le i \le n)$, $r_n \ne 0$, and $r_{i-1} = 0$ whenever $r_i = 2$ $(2 \le i \le n)$. Then

$$\langle u_0 u_1^2 \dots u_n^2, u_0^{1-r_1}u_1^{2-r_2} \cdots u_{n-1}^{2-r_n}u_n^2\rangle = u_0^{r_1}u_1^{r_2} \dots u_{n-1}^{r_n}.$$

Proof: Write $r_1r_2\dots r_n = 0^{s_1}C_10^{s_2}C_2\dots0^{s_m}C_m$, where $s_1\ge 0$, $s_j\ge 1$ $(2\le j\le m)$, each C_j is of the form 1^{t_j}, 2, or 21^{t_j} $(t_j\ge 1)$ and $C_1 = 1^{t_1}$ if $s_1 = 0$. We proceed by induction on m.

Let $m=1$. For simplicity, write s for s_1 and t for t_1. There are four cases according to the values of s and t.

(i) $r_1r_2\dots r_{s+t} = 0^s1^t$ $(s>0, t>0)$:

$$\langle u_0u_1^2\dots u_{s+t}^2,\, u_0u_1^2\dots u_{s-1}^2u_s\dots u_{s+t-1}u_{s+t}^2\rangle$$
$$=\langle u_0u_1^2\dots u_{s-1}^2,\, u_0u_1^2\dots u_{s-1}^2\rangle\langle u_s^2\dots u_{s+t}^2,\, u_s\dots u_{s+t-1}u_{s+t}^2\rangle \text{ [by Lemma 2.2]}$$
$$=u_su_{s+1}\dots u_{s+t-1} \text{ [by (a) and (f) of Lemma 3.1].}$$

(ii) $r_1r_2\dots r_{s+1} = 0^s2$ $(s>0)$:

$$\langle u_0u_1^2\dots u_{s+1}^2,\, u_0u_1^2\dots u_{s\,1}^2u_{s+1}^2\rangle$$
$$=\langle u_0u_1^2\dots u_{s-1}^2,\, u_0u_1^2\dots u_{s-1}^2\rangle\langle u_s^2u_{s+1}^2,\, u_{s+1}^2\rangle \text{ [by Lemma 2.2]}$$
$$=u_s^2 \text{ [by (a) and (d) of Lemma 3.1].}$$

(iii) $r_1r_2\dots r_{s+t+1} = 0^s21^t$ $(s>0, t>0)$:

$$\langle u_0u_1^2\dots u_{s+t+1}^2,\, u_0u_1^2\dots u_{s-1}^2u_{s+1}\dots u_{s+t}u_{s+t+1}^2\rangle$$
$$=\langle u_0u_1^2\dots u_{s-1}^2,\, u_0u_1^2\dots u_{s-1}^2\rangle\langle u_s^2\dots u_{s+t+1}^2,\, u_{s+1}\dots u_{s+t}u_{s+t+1}^2\rangle \text{ [by Lemma 2.2]}$$
$$=u_s^2u_{s+1}\dots u_{s+t} \text{ [by (a) and (g) of Lemma 3.1].}$$

(iv) $r_1r_2\dots r_t = 1^t$ $(t>0)$:

$$\langle u_0u_1^2\dots u_t^2,\, u_1\dots u_{t-1}u_t^2\rangle$$
$$=u_0u_1\dots u_{t-1} \text{ [by (e) of Lemma 3.1].}$$

Thus, the result holds for $m=1$. Now, suppose that the result holds for $m=k$, that is,

$$r_1r_2\dots r_n = 0^{s_1}C_10^{s_2}C_2\dots0^{s_k}C_k, \text{ and}$$
$$\langle u_0u_1^2\dots u_n^2,\, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\rangle = u_0^{r_1}u_1^{r_2}\dots u_{n-1}^{r_n},$$

where $C_1,\dots,C_k$ satisfy the above-mentioned conditions. Let $s_{k+1}\ge 1$ and let $C_{k+1} = 1^t$, 2, or 21^t for some $t\ge 1$. There are three cases to consider:

(i) $C_{k+1} = 1^t$: Let $r_{n+1}r_{n+2}\dots r_p = 0^{s_{k+1}}1^t$, where $p = n+s_{k+1}+t$. Then

$$\langle u_0u_1^2\dots u_p^2,\, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\dots u_{p-t-1}^2u_{p-t}\dots u_{p-1}u_p^2\rangle$$
$$=\langle u_0u_1^2\dots u_n^2,\, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\rangle\langle u_{n+1}^2\dots u_{p-t-1}^2,\, u_{n+1}^2\dots u_{p-t-1}^2\rangle$$
$$\langle u_{p-t}^2\dots u_p^2,\, u_{p-t}\dots u_{p-1}u_p^2\rangle \text{ [by Lemma 2.2]}$$
$$=u_0^{r_1}u_1^{r_2}\dots u_{n-1}^{r_n}\varepsilon u_{p-t}\dots u_{p-1} \text{ [by (a), (f) of Lemma 3.1 and the inductive hypothesis]}$$
$$=u_0^{r_1}u_1^{r_2}\dots u_{p-1}^{r_p}.$$

(ii) $C_{k+1} = 2$: Let $r_{n+1}r_{n+2}\dots r_p = 0^{s_{k+1}}2$, where $p = n+s_{k+1}+1$. Then

$$\langle u_0u_1^2\dots u_p^2,\, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\dots u_{p-2}^2u_p^2\rangle$$
$$=\langle u_0u_1^2\dots u_n^2,\, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\rangle\langle u_{n+1}^2\dots u_{p-2}^2,\, u_{n+1}^2\dots u_{p-2}^2\rangle\langle u_{p-1}^2u_p^2,\, u_p^2\rangle$$

$= u_0^{r_1}u_1^{r_2}\dots u_{n-1}^{r_n}\varepsilon u_{p-1}^2$ [by (a), (d) of Lemma 3.1 and the inductive hypothesis]

$= u_0^{r_1}u_1^{r_2}\dots u_{p-1}^{r_p}$.

(iii) $C_{k+1} = 21^t$: Let $r_{n+1}r_{n+2}\dots r_p = 0^{s_{k+1}}21^t$, where $p = n + s_{k+1} + t + 1$. Then

$\langle u_0u_1^2\dots u_p^2, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\dots u_{p-t-2}^2u_{p-t}\dots u_{p-1}u_p^2\rangle$

$= \langle u_0u_1^2\dots u_n^2, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\rangle\langle u_{n+1}^2\dots u_{p-t-2}^2, u_{n+1}^2\dots u_{p-t-2}^2\rangle$

$\langle u_{p-t-1}^2\dots u_p^2, u_{p-t}\dots u_{p-1}u_p^2\rangle$ [by Lemma 2.2]

$= u_0^{r_1}u_1^{r_2}\dots u_{n-1}^{r_n}\varepsilon u_{p-t-1}^2u_{p-t}\dots u_{p-1}$ [by (a), (g) of Lemma 3.1 and the inductive hypothesis]

$= u_0^{r_1}u_1^{r_2}\dots u_{p-1}^{r_p}$.

This completes the proof.

Proof of Theorem 2.1: (a) Let $m = \sum_{i=1}^n r_iq_{i-1}$ be the generalized Zeckendorf representation of m in the q_i's. Define $r_k = 0$ $(k > n)$. Then

$\langle f, f_m\rangle$

$= \langle u_0u_1^2u_2^2\dots, u_0^{1-r_1}u_1^{2-r_2}u_2^{2-r_3}\dots\rangle$ [by Lemma 2.5]

$= \langle u_0u_1^2\dots u_n^2, u_0^{1-r_1}u_1^{2-r_2}\dots u_{n-1}^{2-r_n}u_n^2\rangle\left\langle \prod_{k=n+1}^{\infty} u_k^2, \prod_{k=n+1}^{\infty} u_k^2\right\rangle$ [by Lemma 2.2]

$= u_0^{r_1}u_1^{r_2}\dots u_{n-1}^{r_n}\varepsilon$ [by Lemma 3.2 and (a) of Lemma 3.1]

$= R(x_{n-1}^{r_n}\dots x_1^{r_2}x_0^{r_1})$ $[u_i = R(x_i),\ i \ge 0]$

$= R(s_m)$ [by Lemma 2.4].

(b) Let $m = \sum_{i=1}^n r_iq_{i-1}$. Then, by Lemmas 2.4-2.5 and the fact that $u_i = R(x_i)$ for all i, we have that

$$\langle u_0u_1^2u_2^2\dots, u_0^{1-r_1}u_1^{2-r_2}u_2^{2-r_3}\dots\rangle = u_0^{r_1}u_1^{r_2}\dots u_{n-1}^{r_n}$$

is another way of writing $\langle f, f_m\rangle = R(s_m)$.

Example: If m is a positive integer having a code 0211020111 with respect to $\sqrt{2}-1$, then $\langle f, f_m\rangle = u_1^2u_2u_3u_5^2u_7u_8u_9$, in view of part (b) of Theorem 2.1. Thus, the extracted word $\langle f, f_m\rangle$ can be found by computing $u_1, u_2, \dots, u_9$. There is no need to compute m, f, and f_m.

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