SUFFIXES OF FIBONACCI WORD PATTERNS

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1. INTRODUCTION

Let $\mathcal{A}$ be an alphabet. Let $\mathcal{A}^*$ be the monoid of all words over $\mathcal{A}$. Let ε denote the empty word, and let $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$. If $w = a_1a_2 \dots a_n$, where $a_i \in \mathcal{A}$, the positive integer n is called the *length* of w, denoted by $|w|$. Let $|\varepsilon| = 0$. A word x is said to be a *prefix* (resp., *suffix*) of w, denoted by $x <_p w$ (resp., $x <_s w$), if there is a word $y \in \mathcal{A}^+$ such that $w = xy$ (resp., $w = yx$). We write $x \le_p w$ (resp., $x \le_s w$) if $x <_p w$ (resp., $x <_s w$) or $x = w$. Prefixes and suffixes of an infinite word are defined similarly.

Let f be an infinite word over $\mathcal{A}$. For $j \ge 0$, let $S^j f$ denote the suffix of f obtained from f by deleting the first j letters of f. For simplicity we write Sf for $S^1 f$. This defines an operator S acting on infinite words over $\mathcal{A}$. The *cyclic shift operator* T on $\mathcal{A}^+$ is given by $T(a_1a_2 \dots a_n) = a_2 \dots a_n a_1$, where $a_i \in \mathcal{A}$. For $j \ge 1$, let $T^j = T(T^{j-1})$, where T^0 denotes the identity operator on $\mathcal{A}^+$. Clearly, each operator T^j has an inverse T^{-j}.

Let $u, v \in \mathcal{A}^+$, $x_1 = u$, $x_2 = v$, and $x_n = x_{n-2}x_{n-1}$ $(n \ge 3)$. The infinite word $x_1x_2x_3 \dots$ is called a *Fibonacci word pattern* generated by u and v and is denoted by $F(u, v)$. The words u and v are called the *seed words* of $F(u, v)$. Let $\mathcal{F}^{m,n}$ denote the set of all Fibonacci word patterns $F(u, v)$ with $|u| = m$ and $|v| = n$. Let $\mathcal{F}$ denote the set of all Fibonacci word patterns.

Given $u, v \in \mathcal{A}^+$, $|u| = m$, $|v| = n$, Turner [17] proved that $F(u, v) \in \mathcal{F}^{r,s}$, where $(r, s) = (F_{2i-1}m + F_{2i}n,\ F_{2i}m + F_{2i+1}n)$ for all $i \ge 1$. In Section 2 of this paper we find necessary and sufficient conditions for $F(u, v)$ to be a member of $\mathcal{F}^{n,m+n}$ (resp., $\mathcal{F}^{n-m,m}$, $\mathcal{F}^{2m-n,n-m}$) (Theorems 2.2-2.4). We also find necessary and sufficient conditions for $SF(u, v)$ to be a member of $\mathcal{F}^{m,n}$ (resp., $\mathcal{F}^{n,m+n}$) (Theorems 2.5-2.6). The fact that $\mathcal{F}$ is invariant under S is a consequence of Theorem 2.7, which asserts that $SF(u, v)$ always belongs to $\mathcal{F}^{m+n,m+2n}$. The Fibonacci word patterns over $\{0, 1\}$ are called *Fibonacci binary patterns* (see [5], [17]). The most famous Fibonacci binary pattern is the *golden sequence* $F(1, 01)$, which is identical to the binary word $c_1c_2 \dots$, where $c_n = [(n+1)\alpha] - [n\alpha]$, $n \ge 1$, and $\alpha = (\sqrt{5} - 1)/2$. See, for example, [2], [3], and [5]-[18]. In Section 3 we use the above results and Lemma 3.1 to compute the possible lengths of the seed words of the suffixes $S^jF(1, 01)$, $j \ge 0$ (Theorem 3.2 and Table 1). It turns out that all these possible pairs of seed words of $S^jF(1, 01)$ have Fibonacci lengths and are pairs of Fibonacci words, the notion of which was introduced by Chuan [4] (see Definition in Section 4). They can be determined by different representations of j in Fibonacci numbers (Theorems 4.5 and 4.6). This gives another proof of Corollary 3.3 of [9] for the case $\alpha = (\sqrt{5} - 1)/2$.

2. FIBONACCI WORD PATTERNS AND THEIR SUFFIXES

Throughout this section, let $u, v \in \mathcal{A}^+$, $|u| = m$, $|v| = n$.

Theorem 2.1 (see [17]): $F(u, v) = F(uv, uvv) \in \mathcal{F}^{m+n,m+2n}$.

Theorem 2.2:

(a) Let $m \le n$. Then $F(u, v) \in \mathcal{F}^{n, m+n}$ if and only if $u \le_s v$. Moreover, $F(u, xu) = F(ux, uux)$ for all $x \in \mathcal{A}^*$.

(b) Let $m > n$, $u = xy$, where $x, y \in \mathcal{A}^+$, $|x| = n$. Then $F(u, v) \in \mathcal{F}^{n, m+n}$ if and only if $xy = yv$. In this case, $F(u, v) = F(x, xyx)$.

Proof: ***(a)*** $(m \le n)$ Suppose that $F(u, v) \in \mathcal{F}^{n, m+n}$. Let $v = xy$, where $x, y \in \mathcal{A}^*$, $|y| = m$. Then

$$F(u, v) = F(u, xy) = (u)(xy)(uxy)(xyuxy)\cdots$$
$$= (ux)(yux)(yxyux)\cdots.$$

Since $F(u, v) \in \mathcal{F}^{n, m+n}$, it follows that

$$F(u, v) = F(ux, yux) = (ux)(yux)(uxyux)\cdots.$$

By comparing the two expressions of $F(u, v)$ and using the assumption that $|y| = |u| = m$, we have $u = y$. This proves that $u \le_s v$, $v = xu$, and $F(u, xu) = F(ux, uux)$.

Conversely, let $v = xu$, where $x \in \mathcal{A}^*$. We claim that $F(u, xu) = F(ux, uux)$. Let

$$x_1 = u, \quad x_2 = v = xu, \quad x_n = x_{n-2}x_{n-1},$$
$$y_1 = ux, \quad y_2 = uux, \quad y_n = y_{n-2}y_{n-1}, \; n \ge 3.$$

Clearly, $u \le_s x_n$, $n \ge 1$. Write $x_n = z_n u$, where $z_n \in \mathcal{A}^*$. Since $x_n = x_{n-2}x_{n-1}$, we have $z_n = z_{n-2}uz_{n-1}$, $n \ge 3$. Now it is easy to see that $y_{n-1} = uz_n$, $n \ge 2$. Therefore,

$$F(u, v) = F(u, xu) = x_1x_2x_3\cdots = u(z_2u)(z_3u)\cdots$$
$$= (uz_2)(uz_3)(uz_4)\cdots = y_1y_2y_3\cdots = F(ux, uux).$$

(b) $(m > n)$ The proof is similar to part (a). □

We note that the condition $xy = yv$ holds if and only if there are words $z_1, z_2 \in \mathcal{A}^*$ and an integer $r \ge 0$ such that $x = z_1z_2$, $y = (z_1z_2)^r z_1$, and $v = z_2z_1$ (see [15]).

Corollary: Let $u \le_s v$ and let $u_k, v_k \in \mathcal{A}^+$ be such that $|u_k| = F_{k-1}m + F_k n$, $|v_k| = F_k m + F_{k+1} n$, and $u_k v_k <_p F(u, v)$, $k \ge 0$. Then $F(u, v) = F(u_k, v_k) \in \mathcal{F}^{|u_k|, |v_k|}$ and $u_k \le_s v_k$. Here $F_{-1} = 1$, $F_0 = 0$.

Theorem 2.3: Let $m < n \le 2m$. Then $F(u, v) \in \mathcal{F}^{n-m, m}$ if and only if u and v have a common prefix of length $n-m$ and $u <_s v$.

Proof: Suppose that $F(u, v) = F(x, z)$, where $|x| = n-m$ and $|z| = m$. It follows from part (a) of Theorem 2.2 that $x \le_s z$, i.e., $z = yx$ for some $y \in \mathcal{A}^*$. Also, $u = xy$ and $v = xxy$. Hence, x is a common prefix of u and v of length $n-m$ and $u <_s v$.

Conversely, suppose that u and v have a common prefix x of length $n-m$ and $u <_s v$. Then $u = xy$, $v = xxy$, where $y \in \mathcal{A}^*$. Then, according to part (a) of Theorem 2.2, we have $F(x, yx) = F(xy, xxy)$. Hence, $F(u, v) \in \mathcal{F}^{n-m, m}$. □

Theorem 2.4 follows from Theorem 2.1.

Theorem 2.4: Let $m < n < 2m$. Then $F(u, v) \in \mathcal{F}^{2m-n, n-m}$ if and only if u and v have a common suffix of length $n-m$ and $u <_p v$.

Theorem 2.5: Let $1 \le k \le \min(m, n)$. Then $S^j F(u, v) \in \mathcal{F}^{m,n}$ for all j, $0 \le j \le k$, if and only if u and v have a common prefix of length k. In this case, $S^j F(u, v) = F(T^j(u), T^j(v))$. If, in addition, $u \le_s v$, then $T^j(u) \le_s T^j(v)$.

Proof: Suppose that $S^k F(u, v) \in \mathcal{F}^{m,n}$. Let $u = wx$, $v = w_1 y$, where w, w_1, x, and y are words and $|w| = |w_1| = k$. Then it is clear that $S^k F(u, v) = F(xw_1, yw)$ and $w = w_1$. Thus, w is a common prefix of both u and v.

Conversely, suppose that u and v have a common prefix az, where $a \in \mathcal{A}$, $z \in \mathcal{A}^*$. Write $u = azx$, $v = azy$, where $x, y \in \mathcal{A}^*$. Then $SF(u, v) = F(zxa, zya) \in \mathcal{F}^{m,n}$. Moreover, z is a common prefix of the seed words zxa, zya of $SF(u, v)$, $|z| = k-1$, $zxa = T(u)$, and $zya = T(v)$. If $u \le_s v$, then clearly $zxa \le_s zya$. Now the result follows by inductive argument. □

The following theorem can be proved in a similar way.

Theorem 2.6:

(a) Let $m \le n$. Then $SF(u, v) \in \mathcal{F}^{n, m+n}$ if and only if u and v have a common suffix of length $m-1$. Moreover, $F(ax, zx) = aF(xz, xaxz)$ for all $a \in \mathcal{A}$, x, $z \in \mathcal{A}^+$.

(b) Let $m > n$, $u = axy$, where $a \in \mathcal{A}$, x, $y \in \mathcal{A}^*$, $|x| = n$. Then $SF(u, v) \in \mathcal{F}^{n, m+n}$ if and only if $xy = yv$. In this case, $F(axy, v) = aF(x, yvax)$.

Corollary: Let $j \ge 0$, u_j, $v_j \in \mathcal{A}^+$, $u_j v_j <_p S^j F(u, v)$, $|u_j| = F_{j-1}m + F_j n$, $|v_j| = F_j m + F_{j+1} n$. If $u \le_s v$, then $S^j F(u, v) = F(u_j, v_j) \in \mathcal{F}^{|u_j|, |v_j|}$ and $u_j \le_s v_j$.

Theorem 2.7: $SF(u, v) \in \mathcal{F}^{m+n, m+2n}$.

Proof: According to Theorem 2.1, $F(u, v) = F(uv, uvv) \in \mathcal{F}^{m+n, m+2n}$. Since uv and uvv have the same first letter, it follows from Theorem 2.5 that $SF(u, v) = SF(uv, uvv) \in \mathcal{F}^{m+n, m+2n}$. □

Corollary: All suffixes of $F(u, v)$ belong to $\mathcal{F}$. More precisely, for $j \ge 0$, $S^j F(u, v) \in \mathcal{F}^{r,s}$, where $(r, s) = (F_{2j-1}m + F_{2j}n, F_{2j}m + F_{2j+1}n)$.

3. THE GOLDEN SEQUENCE $F(1, 01)$

Let $\mathcal{A} = \{0, 1\}$. Consider the golden sequence $f = F(1, 01)$. For each $j \ge 0$, we shall show how to compute pairs of positive integers (r, s) for which $S^j f \in \mathcal{F}^{r,s}$. A key observation is the following lemma.

Lemma 3.1: Let $n \ge 2$ and $F_n - 1 \le j \le F_{n+1} - 2$. Then $S^j f = F(u_j, v_j)$, where $u_j, v_j \in \{0, 1\}^+$, $|u_j| = F_n$, $|v_j| = F_{n+1}$, $u_j <_s v_j$, and u_j, v_j have a common prefix of largest length $F_{n+1} - 2 - j$. (When $n = 2$ and $j = 0$, u_0, v_0 have different first letters.)

Proof: The result clearly holds when $n = 2, 3$. Suppose that it holds for $n = k$. Let $i = F_{k+1} - 2$. It follows from Theorems 2.5 and 2.6 that $S^{i+1} f \in \mathcal{F}^{F_{k+1}, F_{k+2}} \backslash \mathcal{F}^{F_k, F_{k+1}}$. Moreover, $S^{i+1} f = F(u_{i+1}, v_{i+1})$, where $|u_{i+1}| = F_{k+1}$, $|v_{i+1}| = F_{k+2}$, $u_{i+1} <_s v_{i+1}$, and u_{i+1}, v_{i+1} have a common prefix of largest length $F_k - 1$. According to Theorem 2.5, if $1 \le m \le F_k$ and $j = i + m$, then

$S^j f = F(u_j, v_j)$, where $|u_j| = F_{k+1}$, $|v_j| = F_{k+2}$, $u_j <_s v_j$, and u_j, v_j have a common prefix of largest length $F_k - m = F_{k+2} - 2 - j$. Thus, the result holds for all $n \geq 2$. □

Theorem 3.2: Let $n \geq 2$ and $F_n - 1 \leq j \leq F_{n+1} - 2$. Then $S^j f \in \mathcal{F}^{F_k, F_{k+1}}$ if $k \geq n$, and $S^j f \notin \mathcal{F}^{F_k, F_{k+1}}$ if $1 \leq k \leq n-1$.

Proof: The first part is a consequence of Lemma 3.1, Theorem 2.5, and the Corollary to Theorem 2.2. The second part follows from Lemma 3.1 and Theorems 2.1, 2.3, and 2.4. □

For example, when $n = 6$ and $7 \leq j \leq 11$, Theorem 3.2 implies that $S^j f \in \mathcal{F}^{r,s}$, where $(r, s) = (8,13), (13,21), (21,34), \ldots$ and $S^j f \notin \mathcal{F}^{r,s}$, where $(r, s) = (1,2), (2,3), (3,5), (5,8)$. This completes the part of Table 1 corresponding to $7 \leq j \leq 11$.

TABLE 1

j	(r, s) for which $S^j f \in \mathcal{F}^{r,s}$
0	(1,2), (2,3), (3,5), (5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
1	(2,3), (3,5), (5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
2	(3,5), (5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
3	(3,5), (5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
4	(5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
5	(5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
6	(5,8), (8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
7	(8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
8	(8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
9	(8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
10	(8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
11	(8,13), (13,21), (21,34), (34,55), (55,89), (89,144)
12	(13,21), (21,34), (34,55), (55,89), (89,144)
13	(13,21), (21,34), (34,55), (55,89), (89,144)

4. SEED WORDS OF $S^j F(1, 01)$ ARE FIBONACCI WORDS

Again we let $f = F(1, 01)$. We have seen in Theorem 3.2 that, if $n \geq 2$ and $F_n - 1 \leq j \leq F_{n+1} - 2$, then $S^j f \in \mathcal{F}^{F_k, F_{k+1}}$ for all $k \geq n$. Now let (u_{jk}, v_{jk}) denote the pair of seed words of $S^j f$ such that $|u_{jk}| = F_k$ and $|v_{jk}| = F_{k+1}$. We shall show in Theorem 4.5 that u_{jk} and v_{jk} are Fibonacci words, as defined below, whose labels can be determined. Special cases can be found in [5].

Fibonacci words over the alphabet {0, 1} are defined as follows: Let

$$w(0) = 10, \; w(1) = 01,$$
$$w(00) = 101, \; w(01) = 110, \; w(10) = 011, \; w(11) = 101.$$

For any binary sequence $r_1, r_2, \ldots, r_n$, $n \geq 3$, the word $w(r_1 r_2 \ldots r_n)$ is defined recursively by

$$w(r_1 r_2 \ldots r_k) = \begin{cases} w(r_1 r_2 \ldots r_{k-1}) w(r_1 r_2 \ldots r_{k-2}) & \text{if } r_k = 0, \\ w(r_1 r_2 \ldots r_{k-2}) w(r_1 r_2 \ldots r_{k-1}) & \text{if } r_k = 1, \end{cases}$$

$3 \le k \le n$. The word $w(r_1r_2\ldots r_n)$ is called a *Fibonacci word* generated by the pair of words $(0, 1)$. The sequence $r_1, r_2, \ldots, r_n$ is called a *label* of $w(r_1r_2\ldots r_n)$. It describes how the Fibonacci word $w(r_1r_2\ldots r_n)$ is generated. A Fibonacci word may have several different labels. For example, $10101101 = w(0010) = w(1100) = w(1111)$. The words 0 and 1 are also Fibonacci words. For convenience, we write $1 = w(\lambda)$, where λ denotes the empty label. The above notion of Fibonacci word was introduced by Chuan [4] and was later generalized to the notion of α-word by her [8]. :Many known results in the literature involve Fibonacci words (see, e.g., [4]-[12], [16]-[18]).

We need the following properties of Fibonacci words, the proofs of which can be found in [4]. Let $y_1 = 0$, $y_2 = 1$, $y_n = y_{n-2}y_{n-1}$ (i.e., $y_n = w(11\ldots1)$), $n \ge 3$.

Lemma 4.1: Let $n \ge 1$, $r_1, r_2, \ldots, r_n$, $s_1, s_2, \ldots, s_n \in \{0, 1\}$. Then:

(a) $|w(r_1r_2\ldots r_n)| = F_{n+2}$.

(b) If $j = \sum_{i=1}^{n} r_iF_{i+1}$, then $w(r_1r_2\ldots r_n) = T^{-k}(y_{n+2})$, where $k = F_{n+3} - 2 - j$.

(c) If $\sum_{i=1}^{n} s_iF_{i+1} \equiv \sum_{i=1}^{n} r_iF_{i+1} \pmod{F_{n+2}}$, then $w(r_1r_2\ldots r_n) = w(s_1s_2\ldots s_n)$.

Let $u, x \in \mathscr{A}^+$. Then

$$F(u, xu) = F(ux, uux) = uF(xu, uxu) = uxF(uux, uxuux).$$

The first equality follows from part (a) of Theorem 2.2; the second one is trivial; the third one can be proved in a similar way as Theorem 2.2(a). It follows that, if $|u| = m$ and $|x| = t$, then

$$S^mF(u, xu) = F(xu, uxu),$$
$$S^{m+t}F(u, xu) = F(uux, uxuux).$$

In particular, we have the following lemma. Part (d) follows from Theorem 2.1.

Lemma 4.2: Let $n \ge 1$, $r_1, r_2, \ldots, r_n$, $r_{n+1} \in \{0, 1\}$. Let $u = w(r_1r_2\ldots r_n)$, $v = w(r_1r_2\ldots r_n1)$. Then:

(a) $F(u, v) = F(w(r_1r_2\ldots r_n0), w(r_1r_2\ldots r_n01))$.

(b) $S^{F_{n+2}}F(u, v) = F(w(r_1r_2\ldots r_n1), w(r_1r_2\ldots r_n11))$.

(c) $S^{F_{n+3}}F(u, v) = F(w(r_1r_2\ldots r_n01), w(r_1r_2\ldots r_n011))$.

(d) $F(w(r_1r_2\ldots r_n), w(r_1r_2\ldots r_{n+1})) = F(w(r_1\ldots r_{n+1}1), w(r_1\ldots r_{n+1}10))$.

Lemma 4.3 (see [1]): Each positive integer *j* is uniquely expressed as $j = \sum_{i=1}^{n} r_iF_{i+1}$, where $r_n = 1$, $r_i \in \{0, 1\}$, and $\max(r_i, r_{i+1}) = 1$ $(1 \le i \le n-1)$.

The representation $j = \sum_{i=1}^{n} r_iF_{i+1}$ in Lemma 4.3 is called the *maximal representation* of *j*. The code $\langle r_1r_2\ldots r_n\rangle$ is called the *maximal code* of *j*. The number *n* is called the *length* of the maximal code of *j*. For convenience, the maximal code of the integer 0 is defined to be the empty code λ. It has length 0. We note that $F_{n+2} - 1 \le j \le F_{n+3} - 2$ if and only if the length of the maximal code of *j* is *n*.

Lemma 4.4: For each $j \ge 0$, let $\langle r_1r_2\ldots r_n\rangle$ be the maximal code of *j*. Then $S^jf = F(w(r_1r_2\ldots r_n), w(r_1r_2\ldots r_n1))$.

Proof: The result clearly holds for $0 \le j \le 3$. Now suppose that $k > 3$ and that the result is true for all j, $0 \le j < k$. We show that it is also true for $j = k$. Let $n \ge 3$ be such that $F_{n+2} - 1 \le k \le F_{n+3} - 2$.

(a) If $F_{n+2} - 1 \le k \le 2F_{n+1} - 2$, then $F_n - 1 \le k - F_{n+1} \le F_{n+1} - 2$. By the inductive hypothesis,

$$S^{k-F_{n+1}} f = F(w(r_1 r_2 \ldots r_{n-2}), w(r_1 r_2 \ldots r_{n-2} 1)),$$

where $\langle r_1 r_2 \ldots r_{n-2} \rangle$ is the maximal code of $k - F_{n+1}$. Clearly, $\langle r_1 r_2 \ldots r_{n-2} 01 \rangle$ is the maximal code of k. Also,

$$\begin{aligned} S^k f &= S^{F_{n+1}} S^{k-F_{n+1}} f = S^{F_{n+1}} F(w(r_1 r_2 \ldots r_{n-2}), w(r_1 r_2 \ldots r_{n-2} 1)) \\ &= F(w(r_1 r_2 \ldots r_{n-2} 01), w(r_1 r_2 \ldots r_{n-2} 011)), \end{aligned}$$

according to part (c) of Lemma 4.2.

(b) If $2F_{n+1} - 1 \le k \le F_{n+3} - 2$ and if $\langle r_1 r_2 \ldots r_{n-1} \rangle$ is the maximal code of $k - F_{n+1}$, then the inductive hypothesis implies that

$$S^{k-F_{n+1}} f = F(w(r_1 r_2 \ldots r_{n-1}), w(r_1 r_2 \ldots r_{n-1} 1)).$$

Therefore, $\langle r_1 r_2 \ldots r_{n-1} 1 \rangle$ is the maximal code of k and

$$S^k f = F(w(r_1 r_2 \ldots r_{n-1} 1), w(r_1 r_2 \ldots r_{n-1} 11)),$$

according to part (b) of Lemma 4.2. □

Using Lemma 4.4 and part (a) of Lemma 4.2, the seed words of $S^j f$ can now be determined.

Theorem 4.5: Let $j \ge 0$ and let $\langle r_1 r_2 \ldots r_n \rangle$ be the maximal code of j. Let $k \ge n+2$. Then $u_{jk} = w(r_1 r_2 \ldots r_n 0 \ldots 0)$ and $v_{jk} = w(r_1 r_2 \ldots r_n 0 \ldots 01)$ (there are $k - n - 2$ zeros right after r_n).

For example, since $3 = F_2 + F_3$ is the maximal representation of 3, we have $u_{36} = w(1100)$, $v_{36} = w(11001)$. As observed before, the labels for u_{jk} and v_{jk} may not be unique.

Corollary: Let $j \ge 0$ and let n be the smallest integer ≥ 2 such that $j \le F_{n+1} - 2$. If $k \ge n$, then $S^j f = F(T^{-i_k}(y_k), T^{-i_k}(y_{k+1}))$, where $i_k = F_{k+1} - 2 - j$.

Proof: The result follows from Theorem 4.5 and part (b) of Lemma 4.1. □

Note that this corollary contains part (b) of Theorem 8 of [5].

Theorem 4.6: Let $j = \sum_{i=1}^{k-2} s_i F_{i+1}$, where $s_i \in \{0, 1\}$ $(1 \le i \le k-2)$ and $k \ge 3$, then

$$S^j f = F(w(s_1 s_2 \ldots s_{k-2}), w(s_1 s_2 \ldots s_{k-2} 1)).$$

Proof: If $j = 0$, then the result is contained in Theorem 4.5. Now let $j \ge 1$ and let $\langle r_1 r_2 \ldots r_n \rangle$ be the maximal code of j. Clearly, $n \le k-2$. Define $r_i = 0$ if $n < i \le k-2$. Then

$$j = \sum_{i=1}^{k-2} r_i F_{i+1} = \sum_{i=1}^{k-2} s_i F_{i+1},$$

$$j + F_k = \sum_{i=1}^{k-2} r_i F_{i+1} + F_k = \sum_{i=1}^{k-2} s_i F_{i+1} + F_k.$$

Hence,

$$\begin{aligned}(u_{jk}, v_{jk}) &= (w(r_1r_2\dots r_{k-2}), w(r_1r_2\dots r_{k-2}1)) \text{ [by Theorem 4.5]}\\ &= (w(s_1s_2\dots s_{k-2}), w(s_1s_2\dots s_{k-2}1)) \text{ [by part (c) of Lemma 4.1].}\end{aligned}$$

This completes the proof. □

For example, since $3 = F_2 + F_3 = F_4$, we have $u_{36} = w(1100) = w(0010)$ and $v_{36} = w(11001) = w(00101)$. It also follows from Theorem 4.6 that the Fibonacci word pattern generated by a pair of Fibonacci words of the form $w(r_1r_2\dots r_n), w(r_1r_2\dots r_n1)$ is a suffix of f.

Corollary: For every binary sequence $r_1, r_2, \dots, r_n$, the Fibonacci word pattern $F(w(r_1r_2\dots r_n), w(r_1r_2,\dots r_n1))$ is a suffix of f. More precisely,

$$F(w(r_1r_2\dots r_n), w(r_1r_2,\dots r_n1)) = S^j f,$$

where $j = \sum_{i=1}^n r_i F_{i+1}$.

We remark that Theorem 4.6 is a special case of Corollary 3.3 of [9], which was proved by a general representation theorem. In our proof given here, only elementary properties of Fibonacci word patterns and Fibonacci words are used.

Seed words of the Fibonacci word pattern $F(0, 1)$ can also be obtained easily. Let $w_1 = 0$, $w_2 = 1$, and for $n \ge 3$, let $w_n = w_{n-2}w_{n-1}$ if n is odd and $w_n = w_{n-1}w_{n-2}$ if n is even [that is, $w_n = w(r_1r_2\dots r_{n-2})$, where r_i equals 1 if n is odd and equals 0 if n is even $(n \ge 3)$]. It follows immediately from part (d) of Lemma 4.2 that $F(0, 1) = F(w_{2n-1}, w_{2n}) \in \mathcal{F}^{F_{2n-1}, f_{2n}}$ $(n \ge 1)$. Since w_{2n-1} and the suffix of w_{2n} having length $|w_{2n-1}|$ $(= F_{2n-1})$ have different first letters (see [6]), it follows that $F(0, 1) \notin \mathcal{F}^{F_{2n}, F_{2n+1}}$ $(n \ge 1)$, according to part (c) of Theorem 2.2.

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