A COMPOSITE OF GENERALIZED MORGAN-VOYCE POLYNOMIALS

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(Submitted December 1998-Final Revision January 2000)

1. INTRODUCTION

In this note we shall study a class of polynomials $\{R_{n,m}^{(r,u)}(x)\}$, where r and u are integers, n and m are nonnegative integers. The polynomials $\{R_{n,1}^{(r,u)}(x)\}$ and $\{R_{n,2}^{(r,u)}(x)\}$ were studied in [2]. Furthermore, the class of polynomials $\{R_{n,m}^{(r,u)}(x)\}$ involve a great number of known polynomials. Some of these polynomials are (see [2]):

$$\begin{aligned}
&R_{n,1}^{(0,1)}(x) = b_{n+1}(x),\\
&R_{n,1}^{(1,1)}(x) = B_{n+1}(x),\\
&R_{n,1}^{(2,1)}(x) = c_{n+1}(x),\\
&R_{n,1}^{(0,2)}(x) = C_n(x),\\
&R_{n,1}^{(0,0)}(x) = xB_n(x),\\
&R_{n,2}^{(1,0)}(x) = \phi_n(2,-1;x) \ \text{(see [3])},\\
&R_{n,m}^{(r,1)}(x) = P_{n,m}^{(r)}(x) \ \text{(see [4])},
\end{aligned}$$

where $B_n(x)$ and $b_n(x)$ are Morgan-Voyce polynomials (see [1]). In this paper we also consider the sequence of numbers $\{R_{n,3}^{(r,u)}(2)\}$.

2. POLYNOMIALS $\{R_{n,m}^{(r,u)}(x)\}$

First, we define the polynomials $\{R_{n,m}^{(r,u)}(x)\}$ by the following recurrence relation:

$$R_{n,m}^{(r,u)}(x) = 2R_{n-1,m}^{(r,u)}(x) - R_{n-2,m}^{(r,u)}(x) + xR_{n-m,m}^{(r,u)}(x), \ n \ge m, \tag{2.1}$$

with

$$R_{n,m}^{(r,u)}(x) = (n+1)r + u, \ n = 0, 1, \ldots, m-2, \ R_{m-1,m}^{(r,u)}(x) = mr + u + x. \tag{2.2}$$

From (2.1) and (2.2), we get

$$\begin{aligned}
&R_{m,m}^{(r,u)}(x) = u + (m+1)r + (2+u+r)x,\\
&R_{m+1,m}^{(r,u)}(x) = u + (m+2)r + (3+3u+4r)x,\\
&R_{m+2,m}^{(r,u)}(x) = u + (m+3)r + (4+6u+10r)x, \ldots.
\end{aligned} \tag{2.3}$$

Hence, we see that there is a sequence of numbers $\{c_{n,k}^{(r,u)}\}$ such that

$$R_{n,m}^{(r,u)}(x) = \sum_{k=0}^{[(n+1)/m]} c_{n+1,k}^{(r,u)} x^k, \tag{2.4}$$

where $c_{n,k}^{(r,u)} = 0$ for $k < 0$ or $k > [(n+1)/m]$.

If we take $x=0$ in (2.4), then we have

$$R_{n,m}^{(r,u)}(0)=c_{n+1,0}^{(r,u)}. \tag{2.5}$$

Furthermore, from (2.1), (2.2), and (2.4), we get

$$c_{n+1,k}^{(r,u)}=2c_{n,0}^{(r,u)}-c_{n-1,0}^{(r,u)},\ n\geq 1, m\geq 2, \tag{2.6}$$

with

$$c_{0,0}^{(r,u)}=u+r,\ c_{1,0}^{(r,u)}=u+2r. \tag{2.7}$$

The solution of the difference equation (2.6), using (2.7), is

$$c_{n,0}^{(r,u)}=u+(n+1)r,\ n\geq 0. \tag{2.8}$$

Again from (2.1) and (2.4), we have

$$c_{n,k}^{(r,u)}=2c_{n-1,k}^{(r,u)}-c_{n-2,k}^{(r,u)}+c_{n-m,k-1}^{(r,u)},\ k\geq 1, n\geq m. \tag{2.9}$$

3. COEFFICIENTS $c_{n,k}^{(r,u)}$

The main purpose in this section is to determinate the coefficients $\{c_{n,k}^{(r,u)}\}$ for $k\geq 1$. First of all, we shall write the coefficients $\{c_{n,k}^{(r,u)}\}$ in the following form:

TABLE 1

$n\backslash k$	0	1	2	$\ldots$
0	$r+u$	$\cdots$	$\cdots$	$\cdots$
1	$2r+u$	$\cdots$	$\cdots$	$\cdots$
2	$3r+u$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	$mr+u$	1	$\cdots$	$\cdots$
m	$(m+1)r+u$	$2+u+r$	$\cdots$	$\cdots$
$m+1$	$(m+2)r+u$	$3+3u+4r$	$\cdots$	$\cdots$
$m+2$	$(m+3)r+u$	$4+6u+10r$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Now we shall prove the following theorem, using induction.

Theorem 3.1: The coefficients $c_{n,k}^{(r,u)}$ are given by

$$c_{n+1,k}^{(r,u)}=u\binom{n-(m-2)k}{2k}+r\binom{n+1-(m-2)k}{2k+1}+\binom{n-(m-2)k}{2k-1}, \tag{3.1}$$

where $n\geq 0$ and $0\leq k\leq[(n+1)/m]$.

Proof: For $n=0,1,\ldots,m-2$, we obtain $k=0$. Then, from (2.8), we see that (3.1) is true. We shall assume that (3.1) is true for $n\ (n\geq 1)$. Then, by (2.9) and (3.1), we get

$$\begin{aligned}c_{n,k}^{(r,u)}&=2c_{n-1,k}^{(r,u)}-c_{n-2,k}^{(r,u)}+c_{n-m,k-1}^{(r,u)}\\&=\alpha_{n,k}+r\beta_{n,k}+u\gamma_{n,k},\end{aligned} \tag{3.2}$$

where

$$\alpha_{n,k} = 2\binom{n-1-(m-2)k}{2k-1} - \binom{n-2-(m-2)k}{2k-1} + \binom{n-m-(m-2)(k-1)}{2k-3},$$
$$\beta_{n,k} = 2\binom{n-(m-2)k}{2k+1} - \binom{n-1-(m-2)k}{2k+1} + \binom{n+1-m-(m-2)(k-1)}{2k-1}, \tag{3.3}$$
$$\gamma_{n,k} = 2\binom{n-1-(m-2)k}{2k} - \binom{n-2-(m-2)k}{2k} + \binom{n-m-(m-2)(k-1)}{2k-2}.$$

Using the known equality $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$, by equalities (3.3) we have

$$\alpha_{n,k} = \binom{n-(m-2)k}{2k-1}, \; \beta_{n,k} = \binom{n+1-(m-2)k}{2k+1}, \; \gamma_{n,k} = \binom{n-(m-2)k}{2k}. \tag{3.4}$$

Hence, (3.1) is true for all $n \geq 0$.

Corollary 3.1: If $m=1$, then (3.1) becomes (see [2], eq. (2.11)):

$$c_{n+1,k}^{(r,u)} = \binom{n+k}{2k-1} + r\binom{n+k+1}{2k+1} + u\binom{n+k}{2k}.$$

Corollary 3.2: If $m=2$ and n is even, then by (3.1) we have (see [2], eq. (6.3)):

$$c_{n/2+1,\,n/2}^{(r,u)} = u+r+n. \tag{3.5}$$

Using the standard methods, from (2.1) and (2.2), we can prove the following theorem.

Theorem 3.2: The polynomials $\{R_{n,m}^{(r,u)}(x)\}$, for $m \geq 2$, possess the following generating function:

$$\sum_{n=0}^{\infty} R_{n,m}^{(r,u)}(x)t^n = \frac{u+r-ut+xt^{m-1}}{1-2t+t^2-xt^m}. \tag{3.6}$$

Corollary 3.3: For $m=2$ in (3.6), we get the generating function of the polynomials $\{R_n^{(r,u)}(x)\}$,

$$\sum_{n=0}^{\infty} R_n^{(r,u)}(x)t^n = \frac{u+r+(x-u)t}{1-2t+t^2(1-x)}, \tag{3.7}$$

(see [2]).

The Binet Form for $R_{n,3}^{(r,u)}(x)$

For $m=3$ in (2.1) and (2.2), we get the polynomials $\{R_{n,3}^{(r,u)}(x)\}$ such that

$$R_{n,3}^{(r,u)}(x) = 2R_{n-1,3}^{(r,u)}(x) - R_{n-2,3}^{(r,u)}(x) + xR_{n-3,3}^{(r,u)}(x), \; n \geq 3, \tag{3.8}$$

with

$$R_{0,3}^{(r,u)}(x) = r+u, \; R_{1,3}^{(r,u)}(x) = 2r+u, \; R_{2,3}^{(r,u)}(x) = 3r+u+x. \tag{3.9}$$

Using the known methods, by (3.8) and (3.9), we find the Binet form for $\{R_{n,3}^{(r,u)}(x)\}$. That is, we can prove the following theorem.

Theorem 3.3: The Binet form for $\{R_{n,3}^{(r,u)}(x)\}$ is given by

$$R_{n,3}^{(r,u)}(x) = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n, \tag{3.10}$$

where

$$\begin{aligned}\lambda_1 &= \frac{2}{3} + \frac{1}{3}(\alpha(x) + \beta(x)),\\ \lambda_2 &= \frac{2}{3} + \frac{\varepsilon_1}{3}(\alpha(x) + \varepsilon_1\beta(x)),\\ \lambda_3 &= \frac{2}{3} + \frac{\varepsilon_1}{3}(\varepsilon_1\alpha(x) + \beta(x)),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\alpha(x) &= \left(\frac{\Delta(x) + \sqrt{\Delta(x)^2 - 4}}{2}\right)^{1/3},\\ \beta(x) &= \left(\frac{\Delta(x) - \sqrt{\Delta(x)^2 - 4}}{2}\right)^{1/3},\\ \Delta(x) &= 27x - 2,\\ \varepsilon_1 &= \frac{-1 + i\sqrt{3}}{2},\end{aligned} \tag{3.12}$$

for $(\Delta(x))^2 - 4 \geq 0$.

The coefficients C_1, C_2, and C_3 are the solutions of the equation

$$\lambda^3 - 2\lambda^2 + \lambda - x = 0, \tag{3.13}$$

with starting values (3.9). Namely, we get

$$C_i = \frac{\lambda_i x + (r+u)(\lambda_i^2 + x - \lambda_i) + r\lambda_i^2}{2x + \lambda_i^3 - \lambda_i}, \; i = 1, 2, 3. \tag{3.14}$$

4. SEQUENCE OF NUMBERS

The sequence of numbers $\{R_{n,3}^{(r,u)}(1)\}$ was studied in [2]. In this section we shall consider the sequence of numbers $\{R_{n,3}^{(r,u)}(2)\}$. Namely, for $m = 3$ and $x = 2$, from (2.1) and (2.2), we get the following difference equation,

$$a_{n+3} = 2a_{n+2} - a_{n+1} + 2a_n, \; n \geq 0, \tag{4.1}$$

with

$$a_0 = r + u, \; a_1 = 2r + u, \; a_2 = 3r + u + 2, \tag{4.2}$$

where $R_{n,3}^{(r,u)}(2) \equiv a_n$.

The characteristic equation for (4.1) is

$$\lambda^3 - 2\lambda^2 + \lambda - 2 = 0, \tag{4.3}$$

whose roots are

$$\lambda_1 = 2, \; \lambda_2 = i, \; \lambda_3 = -i, \tag{4.4}$$

with

$$\lambda_1 + \lambda_2 + \lambda_3 = 2, \; \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 1, \; \lambda_1\lambda_2\lambda_3 = 2. \tag{4.5}$$

Hence, we have the following representation:

$$a_n = C_1 \cdot 2^n + C_2 \cdot i^n + C_3 \cdot (-i)^n. \tag{4.6}$$

From (4.6) and (4.2), we get the following system of linear equations:

$$C_1+C_2+C_3=r+u,$$
$$2C_1+i\cdot C_2-i\cdot C_3=2r+u,$$
$$4C_1-C_2-C_3=3r+u+2,$$

whose solutions are

$$\begin{aligned} C_1&=\frac{2}{5}(2r+u+1),\\ C_2&=\frac{1}{10}(r+3u-2-i(2r+u-4)),\\ C_3&=\frac{1}{10}(r+3u-2+i(2r+u-4)). \end{aligned} \tag{4.7}$$

Hence, from (4.6), it follows that

$$a_n=\frac{2^{n+1}}{5}(2r+u+1)+\frac{i^n}{10}(r+3u-2-i(2r+u-4))+\frac{(-i)^n}{10}(r+3u-2+i(2r+u-4)). \tag{4.8}$$

Using (4.8), we find that

$$\begin{aligned} a_{4n}&=\frac{1}{5}(2^{4n+1}(2r+u+1)+r+3u-2),\\ a_{4n+1}&=\frac{1}{5}(2^{4n+2}(2r+u+1)+2r+u-4),\\ a_{4n+2}&=\frac{1}{5}(2^{4n+3}(2r+u+1)-r-3u+2),\\ a_{4n+3}&=\frac{1}{5}(2^{4(n+1)}(2r+u+1)-2r-u+4). \end{aligned} \tag{4.9}$$

Remark: It is interesting to consider a generalized numerical sequence $\{R_{n,m}(2^{m-2})\}$, $m\geq 2$, For example, if $m=4$, we have

$$R_{n,4}(4)=C_1(-1)^n+C_2 2^n+C_3\left(\frac{1+i\sqrt{7}}{2}\right)^n+C_4\left(\frac{1-i\sqrt{7}}{2}\right)^n,$$

where the coefficients C_i, $i=1,2,3,4$, are the solutions of the following system:

$$C_1+C_2+C_3+C_4=r+u,$$
$$-C_1+2C_2+\frac{1+i\sqrt{7}}{2}C_3+\frac{1-i\sqrt{7}}{2}C_4=2r+u,$$
$$C_1+4C_2+C_3\left(\frac{1+i\sqrt{7}}{2}\right)^2+C_4\left(\frac{1-i\sqrt{7}}{2}\right)^2=3r+u,$$
$$-C_1+8C_2+C_3\left(\frac{1+i\sqrt{7}}{2}\right)^3+C_4\left(\frac{1-i\sqrt{7}}{2}\right)^3=4(r+1)+u.$$

ACKNOWLEDGMENT

The author is very grateful to the anonymous referee for helpful comments and suggestions concerning the presentation of this paper.

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AMS Classification Numbers: 11B39, 26A24, 11B83

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LETTER TO THE EDITOR

February 14, 2000

Professor Cooper,

I would like to bring your attention to an error in Paul S. Bruckman's article entitled "On the Degree of the Characteristic Polynomial of Powers of Sequences," *The Fibonacci Quarterly* **38.1** (2000):35-38. In particular, the following counterexample illustrates the error. In the notation of the article, let $U_n = 1 + 2^n + 4^n$ with the characteristic polynomial $P_1(z) = (z-1)(z-2)(z-4)$ of degree $R_1 = 3$ with $m = 3$ roots. The main theorem then predicts

$$R_3 = \binom{5}{3} = 10.$$

However, on expanding $(U_n)^3$ we get $(U_n)^3 = 1 + 3 * 2^n + 6 * 4^n + 7 * 8^n + 6 * 16^n + 3 * 32^n + 64^n$ with a characteristic polynomial of degree only 7, namely, $P_3(z) = (z-1)(z-2)(z-4)(z-8)(z-16)(z-32)(z-64)$.

The particular reasoning error in Bruckman's article revolves around the assumption that the products of the powers of the original eigenvalues are all distinct, indicated implicitly in the equation for $P_k(z)$ right before equation (7) on page 36 of the article. I brought attention to this issue in my recent article in your quarterly, noting that *if each root has a unique prime divisor that distinguishes it from the other roots*, then the final order of the power $(U_n)^k$ can be determined easily.

Regrettably, the general result stated in Bruckman's paper is erroneous.

For your careful consideration.

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