r-GENERALIZED FIBONACCI SEQUENCES AND THE LINEAR MOMENT PROBLEM

Bouazza El Wahbi
Département de Mathématiques, Faculté des Sciences
Université Abdelmalek Essadi, B.P. 2121, Tétouan-Morocco

Mustapha Rachidi
Département de Mathématiques et Informatique, Faculté des Sciences
Université Mohammed V, B.P. 1014, Rabat-Morocco

(*Submitted November 1997-Final Revision April 2000*)

1. INTRODUCTION

Let $a_0, a_1, \ldots, a_{r-1}$ $(r \geq 2)$ be real numbers with $a_{r-1} \neq 0$. An r-generalized Fibonacci sequence $\{V_n\}_{n\geq 0}$ is defined by the initial conditions $(V_0, V_1, \ldots, V_{r-1})$ and the following linear recurrence relation of order r,

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_{r-1} V_{n-r+1} \quad \text{for } n \geq r-1. \tag{1}$$

In the sequel we shall refer to such sequences as *sequences* (1). When $a_j = 1$ for all j $(0 \leq j \leq r-1)$ and $V_0 = \cdots = V_{r-2} = 0$, $V_{r-1} = 1$, sequence (1) defines the well-known r-generalized Fibonacci numbers introduced by Miles in [9], which have been studied extensively in the literature. Let $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ be the characteristic polynomial of sequence (1) and set $\sigma_P = \{\lambda \in \mathbf{C};\ P(\lambda) = 0\}$.

Let $(E, (.,.))$ be a unitary real vector space of finite dimension m, and consider $\Lambda(E)$, the space of linear self-adjoint operators on E. An operator $S \in \Lambda(E)$ is called *simple* if its spectrum $\sigma(S) = \{\mu_1, \mu_2, \ldots, \mu_m\}$ is such that $\mu_i \neq \mu_j$ for $i \neq j$. Set $\Lambda_s(E) = \{S \in \Lambda(E);\ S \text{ is simple}\}$ and $\Lambda_s^P(E) = \{S \in \Lambda_s(E);\ \sigma(S) \cap \sigma_P \neq \emptyset\}$. For any $S \in \Lambda(E)$, the sequence $\{V_n\}_{0\leq n\leq p}$ defined by $V_n = (S^n x, x)$ for $n = 0, 1, \ldots, p \leq \infty$, where $x \neq 0$ is a vector of E, is called a *sequence of moments* of S on the vector x. The *linear moment problem* associated to a sequence $\{V_n\}_{0\leq n\leq p}$ consists of finding $S \in \Lambda_s(E)$ such that

$$V_n = (S^n x, x) \quad \text{for } n = 0, 1, \ldots, p \leq \infty, \tag{2}$$

where $x \neq 0$ is a vector of E (see [5] and [6]). In expression (2), the vector x is considered as fixed because it is not unique.

The aim of this paper is to study the linear moment problem for a given sequence (1). More precisely, we give a necessary and sufficient condition for the sequence (1) to be a sequence of moments of $S \in \Lambda_s(E)$. Applications and examples are given. In particular, we can characterize sequences (1) which are linear combinations of geometric sequences. We also consider an application to the study of a linear system of Vandermonde type.

2. LINEAR MOMENT PROBLEM FOR SEQUENCES (1)

2.1 Moments of an Operator and Sequences (1)

Let $(E, (.,.))$ be a unitary real vector space of dimension m and $S \in \Lambda_s(E)$ such that $\sigma(S) = \{\lambda_1, \dots, \lambda_m\}$. We have $E = \oplus_{j=1}^m E_j$, where E_j is the eigenspace $E_j = \{x \in E;\ Sx = \lambda_j x\}$. Let $\{e_1, e_2, \dots, e_m\}$ be an orthogonal basis of E, where $e_j \in E_j$. Set $\{V_n\}_{n\geq 0}$, the sequence of moments of S on a fixed (nonvanishing) vector $x = \sum_{j=1}^m \mu_j e_j$ of E. Let $P_c(X) = \prod_{j=1}^m (X - \lambda_j)$ be the characteristic polynomial of S, and consider a polynomial $Q(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ such that $P_c(X)$ is a divisor of $Q(X)$; we derive from the Cayley-Hamilton theorem that $Q(S) = 0$, and then $S^{n+1} = a_0 S^n + \cdots + a_{r-1} S^{n-r+1}$ for any $n \geq r-1$. Thus, the sequence of moments $V_n = (S^n x, x)$ $(n \geq 0)$ of S on x is a sequence (1). If $r \leq m-1$, we suppose that $\sigma(S) \cap \sigma_Q = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and set $S_1 = S_{|L}$, where $L = \oplus_{j=1}^k E_j$. It is clear that $S_1 \in \Lambda_s(L)$ and $Q(S_1) = 0$. Then, for any $x \in L$ (with $x \neq 0$), the sequence of moments $V_n = (S_1^n x, x)$ $(n \geq 0)$ is again a sequence (1).

In the sequel, we study the converse question. More precisely, we study the linear moment problem (2) for a given sequence (1).

2.2 Reduction of the Linear Moment Problem for Sequences (1)

Let $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ be the characteristic polynomial of sequence (1) and let $\sigma_p = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ be the set of characteristic roots of the sequence (1).

Lemma 2.1: Let $\{V_n\}_{n\geq 0}$ be a sequence (1). Suppose that there exists $S \in \Lambda_s(E)$ such that $V_n = (S^n x, x)$ for all $n \geq 0$, where $x \neq 0$ is a (fixed) vector of E. Then $\sigma(S) \cap \sigma_P = \{\lambda_1, \dots, \lambda_l\} \neq \emptyset$ and $x = \sum_{j=1}^l x_j$, where $x_j \in E_j$.

Proof: Let $S \in \Lambda_s(E)$ and, for any $n \geq r-1$, set $R_n = S^{n+1} - a_0 S^n - \cdots - a_{r-1} S^{n-r+1}$. Then we have $R_n x = \sum_{j=1}^m \mu_j \lambda_j^{n-r+1} P(\lambda_j) e_j$ for any $x = \sum_{j=1}^m \mu_j e_j \in E$. Using equation (2), we obtain the following system of m linear equations in the unknown variables $\mu_1^2, \mu_2^2, \dots, \mu_m^2$,

$$\sum_{j=1}^m \lambda_j^k P(\lambda_j) \mu_j^2 = 0;\ k = 0, 1, \dots, m-1,$$

by taking $n = r-1, r, \dots, r+m-2$. The determinant of this system of Vandermonde type is $\Delta = P(\lambda_1) \dots P(\lambda_m) \prod_{1 \leq i < j \leq m} (\lambda_j - \lambda_i)$. The operator S is simple, so $\lambda_j \neq \lambda_i$ for $i \neq j$, and because $x \neq 0$ we must have $\Delta = 0$, which implies that $\sigma(S) \cap \sigma_P = \{\lambda_1, \dots, \lambda_l\} \neq \emptyset$ and $x = \sum_{j=1}^l \mu_j e_j$. □

If $P(X)$ does not have a real root, Lemma 2.1 shows that the sequence (1) is not a sequence of moments of an operator $S \in \Lambda_s^P(E)$. Let $S \in \Lambda_s(E)$; if $\sigma(S) \cap \sigma_P = \emptyset$, then the sequence (1) cannot be a sequence of moments of the operator S. A partial converse of Lemma 2.1 is given by Lemma 2.2.

Lemma 2.2: Let $\{V_n\}_{n\geq 0}$ be a sequence (1). Suppose that there exists $S \in \Lambda_s^P(E)$ with $\sigma(S) \cap \sigma_P = \{\lambda_1, \dots, \lambda_l\}$. Then there exists a vector $x \neq 0$ in E such that $(S^{n+1}x, x) = \sum_{j=0}^{r-1} a_j (S^{n-j}x, x)$ for all $n \geq r-1$. More precisely, we have $x = \sum_{j=1}^l x_j$, where $x_j \in E_j$.

Proof: Let $S \in \Lambda_s(E)$ and set $\sigma(S) = \{\lambda_1, \ldots, \lambda_m\}$. Then we have the orthogonal decomposition $E = \oplus_{j=1}^m E_j$, where $E_j = \{x \in E; Sx = \lambda_j x\}$. Suppose that $\sigma(S) \cap \sigma_P = \{\lambda_1, \ldots, \lambda_s\}$. For any λ_j $(0 \le j \le s)$ and $x_j \in E_j$, we have $S^k x_j = \lambda_j^k x_j$ $(k \ge 0)$ and $\lambda_j^{k+1} = a_0 \lambda_j^k + \cdots + a_{r-2}\lambda_j^{k-r+2} + a_{r-1}\lambda_j^{k-r+1}$ $(k \ge r-1)$. Thus, we have $(S^{n+1}x_j, x_j) = \sum_{j=0}^{r-1} a_j (S^{n-j}x_j, x_j)$ for all $n \ge r-1$. Because the decomposition $E = \oplus_{j=1}^m E_j$ is orthogonal, we derive $(S^{n+1}x, x) = \sum_{j=0}^{r-1} a_j (S^{n-j}x, x)$ for any $n \ge r-1$, where $x = \sum_{j=1}^{l} x_j$ $(x_j \in E_j)$. □

Example 2.1—Characterization of geometric r-generalized Fibonacci sequences which are sequences of moments: If $E = \mathbf{R}$, a simple self-adjoint operator S on E is defined by $S(x) = \lambda x$, where $\lambda = S(1)$ and $\sigma(S) = \{\lambda\}$. Let $\{V_n\}_{n\ge0}$ be a sequence (1). If $V_n = (S^n x, x)$ for all $n \ge 0$, we derive that $x^2 = V_0$ and $V_n = (V_1/V_0)^n V_0$, $n = 1, 2, \ldots, r-1$. For $n \ge r$, expression (1) allows us to have $V_1^r = \sum_{j=0}^{r-1} a_j V_0^{j+1} V_1^{r-j-1}$. Then $\{V_n\}_{n\ge0}$ is a sequence of moments of $S \in \Lambda_s(\mathbf{R})$ on $x \ne 0$ if and only if $x^2 = V_0$, $V_n = (V_1/V_0)^n V_0$, $n = 1, 2, \ldots, r-1$, and $Q(V_0, V_1)=0$, where $Q(X, Y) = Y^r - \sum_{j=0}^{r-1} a_j X^{j+1} Y^{r-j-1}$. Geometrically, sequence (1) is a sequence of moments of $S \in \Lambda_s(\mathbf{R})$ on $x \ne 0$ if and only if $x^2 = V_0$, $V_n = (V_1/V_0)^n V_0$, $n = 1, 2, \ldots, r-1$, and (V_0, V_1) is a point of the algebraic curve of equation $Q(X, Y) = 0$. □

A subspace L of E is called *invariant* under $S \in \Lambda(E)$ (or *S-invariant*) if $Sx \in L$ for all $x \in L$, and we denote by $S_{|L}$ the restriction of S to L.

Lemma 2.3: Let $S \in \Lambda_s(E)$ and let L be a nontrivial S-invariant subspace of E. Then $P(S_{|L}) = 0$ if and only if $L \subseteq \oplus_{\lambda \in \sigma(S)\cap\sigma_P} E_\lambda$.

The proof of this lemma my be deduced using the fact that any operator $S \in \Lambda_s(E)$ defines a basis of eigenvectors of E and its restriction to any nontrivial S-invariant subspace L is an operator of $\Lambda_s(L)$.

From Lemmas 2.1, 2.2, and 2.3, we can derive the following proposition.

Proposition 2.1: Let $\{V_n\}_{n\ge0}$ be a sequence (1). Suppose that there exists $S \in \Lambda_s(E)$ such that $V_n = (S^n x, x)$ for all $n \ge 0$, where $x \ne 0$ is a vector of an S-invariant subspace L of E. Then we have $V_n = (S_{|L}^n x, x)$ for any $n \ge 0$.

With the aid of Lemma 2.2 and Proposition 2.1, the linear moment problem for a sequence (1) may be reduced as follows: Find $S \in \Lambda_s(E)$ such that $\sigma(S) \cap \sigma_P \ne \emptyset$ and $V_n = (S^n x, x)$ for $n = 0, 1, \ldots, r-1$, with $x \ne 0$ in $L = \oplus_{\lambda \in \sigma(S)\cap\sigma_P} E_\lambda$, where $E_\lambda = \{x \in E; Sx = \lambda x\}$. Thus, from Lemmas 2.1-2.2 and Proposition 2.1, we derive the following result.

Theorem 2.1: Let $\{V_n\}_{n\ge0}$ be a sequence (1). Then $\{V_n\}_{n\ge0}$ is a sequence of moments of $S \in \Lambda_s(E)$ on a vector $x \ne 0$ of E if and only if $S \in \Lambda_s^P(E)$ and S is a solution of the reduced moment problem

$$V_n = (S^n x, x) \quad \text{for } n = 0, 1, \ldots, r-1, \tag{3}$$

where $x \in L = \oplus_{\lambda \in \sigma(S)\cap\sigma_P} E_\lambda$.

Suppose the reduced linear moment problem (3) has a solution $S \in \Lambda_s(E)$ with $x \ne 0$ in L, an S-invariant subspace of E. If $P(S_{|L}) = 0$, then S is a solution of the linear moment problem (2).

Proposition 2.2: Let $\{V_n\}_{n\geq 0}$ be a sequence (1). Suppose that $S \in \Lambda_s(E)$ is a solution of the linear moment problem (3) with $x \neq 0$ in L, an S-invariant subspace of E, such that $P(S_{|L}) = 0$. Then any extension $S_1 \in \Lambda_s(E)$ of $S_{|L}$ to E satisfies $V_n = (S_1^n x, x)$ for all $n \geq 0$.

Example 2.2: Let $\{V_n\}_{n\geq 0}$ be a sequence (1) with $V_0 > 0$ and $S \in \Lambda_s(E)$. Suppose $\sigma(S) \cap \sigma_P = \{\lambda\}$; it is obvious that $\lambda \neq 0$ because $a_{r-1} \neq 0$. Let e_1 be a basis of E_λ with $(e_1, e_1) = 1$. Then $\{V_n\}_{n\geq 0}$ is a sequence of moments of S on $x = \sqrt{V_0}$ (or $-\sqrt{V_0}$) if and only if $V_k = (S^k x, x) = \lambda^k V_0$ for any $k = 0, 1, \ldots, r-1$. This example is an extension of Example 2.1. Thus, we have the same geometrical interpretation. □

Example 2.3: Let $(E, (.,.))$ be a unitary real vector space of dimension m and let $\{V_n\}_{n\geq 0}$ be a sequence (1) with $r \geq 2$. Suppose that the reduced linear moment problem, $V_k = (S^k x, x)$ for $k = 0, 1, \ldots, r-1$, where $x \neq 0$, has a solution $S \in \Lambda_s(E)$. Let $\sigma(S) \cap \sigma_P = \{\lambda_1, \lambda_2\}$, with $\lambda_1 < \lambda_2$. Let e_j be a basis of E_j with $(e_j, e_j) = 1$ $(j = 1, 2)$. It is obvious that $(e_1, e_2) = 0$. Then we have $V_k = (S^k x, x) = \lambda_1^k a^2 + \lambda_2^k b^2$ for any $k \geq 0$, where $x = ae_1 + be_2$. If $r = 2$, we have

$$a^2 = \frac{\lambda_2 V_0 - V_1}{\lambda_2 - \lambda_1} > 0 \quad \text{and} \quad b^2 = \frac{V_1 - \lambda_1 V_0}{\lambda_2 - \lambda_1} > 0.$$

If $r \geq 3$, we have

$$a^2 = \frac{1}{\lambda_2^{k-1}} \frac{\lambda_2 V_{k-1} - V_k}{\lambda_2 - \lambda_1} > 0 \quad \text{and} \quad b^2 = \frac{1}{\lambda_2^{k-1}} \frac{V_k - \lambda_1 V_{k-1}}{\lambda_2 - \lambda_1} > 0$$

for any $k = 1, \ldots, r-1$. These expressions imply that

$$V_k = \frac{1}{\lambda_2 - \lambda_1}[(\lambda_2 V_0 - V_1)\lambda_1^k + (V_1 - \lambda_1 V_0)\lambda_2^k]$$

for all $k \geq 0$. □

2.3 Sequences (1) and Associated Matrix S

For the construction of $S \in \Lambda_s(E)$ associated to a given sequence (1), it is more convenient to consider a unitary real vector space $(E, (.,.))$ of dimension $m = card\{\lambda_j \in \mathbf{R} \cap \sigma_P\} \leq r$. In this case, we set $\sigma(S) \cap \sigma_P = \{\lambda_1, \ldots, \lambda_l\} \subset \mathbf{R}$ and consider an orthogonal basis $\{e_1, \ldots, e_m\}$ of E, where $Se_j = \lambda_j e_j$ for $j = 1, \ldots, l$. Then S may be identified with the diagonal matrix $D = diag(\lambda_1, \ldots, \lambda_l)$. If $m \geq r+1$, Theorem 2.1 and Proposition 2.2 allow us to see that we can consider a self-adjoint extension S_1 of S and $x \neq 0$, $x \in L = \oplus_{\lambda \in \sigma(S_1) \cap \sigma_P} E_\lambda$.

3. REDUCED LINEAR MOMENT PROBLEM OF SEQUENCES (1) AND HANKEL FORMS

3.1 Hankel Matrices and Hankel Forms

A real (or complex) matrix $M = (a_{jk})_{0 \leq j, k \leq p}$, where $0 \leq p \leq +\infty$, is called *positive semi-definite* (resp. *positive definite*) if $\sum_{0 \leq j, k \leq m} a_{jk} \eta_j \overline{\eta}_k \geq 0$ (resp. > 0) for any finite sequence $\eta = \{\eta_j\}_{0 \leq j \leq m}$, where $\overline{z}$ denotes the complex conjugate of z. Let $\gamma = \{\gamma_j\}_{j \geq 0}$ be a sequence of real or complex numbers. The family of matrices defined by $H(m) = (\gamma_{j+k})_{0 \leq j, k \leq m-1}$, where $m = 1, 2, \ldots$, are called *Hankel matrices* associated with $\gamma = \{\gamma_j\}_{j\geq 0}$, and the family of quadratic forms defined by $\mathcal{H}_m(\eta, \eta) = \sum_{0\leq j,k\leq m-1} \gamma_{j+k}\eta_j\overline{\eta}_k$, where $\eta = \{\eta_j\}_{0\leq j\leq m-1}$, are called *Hankel forms*. An infinite Hankel matrix, $H = (\gamma_{j+k})_{j,k\geq 0}$, is called *positive semidefinite* (resp. *positive definite*) if, for any m, the Hankel form $\mathcal{H}_m$ is positive semidefinite (resp. positive definite) or, equivalently, the Hankel matrices $H(m) = (\gamma_{j+k})_{0\leq j,k\leq m-1}$ $(m = 1, 2, \ldots)$ are positive semidefinite (resp. positive definite). Hankel matrices and forms play an important role in the theory of moment problems (see, e.g., [1]-[6]).

3.2 Linear Moment Problem of Sequences (1) and Hankel Forms

Let $(E, (.,.))$ be a unitary real vector space of finite dimension m and fix an orthogonal basis $\{e_1, e_2, \ldots, e_m\}$ of E. Let $A = (V_0, \ldots, V_{r-1})$ $(r \geq 2)$ be a sequence of real numbers, and consider the real quadratic Hankel forms on E defined by $\mathcal{H}_p^A(x, x) = \sum_{0\leq j,k\leq p-1} V_{j+k}\xi_j\xi_k$ $(p \geq 1)$ for $x = \sum_{j=1}^m \xi_j e_j$. Suppose $r = 2m-1$ and that the Hankel form $\mathcal{H}_m^A$ is positive definite, and consider the scalar product on the $\mathbf{K}$-vector space $\mathbf{K}_{m-1}[X]$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) of polynomials of degree $\leq m-1$, defined by $(P, Q) = \sum_{0\leq j,k\leq m-1} \gamma_{j+k}\zeta_j\overline{\eta}_k$, where $P = \sum_{0\leq j\leq m-1} \zeta_j X^j$ and $Q = \sum_{0\leq j\leq m-1} \eta_j X^j$. Let $S: \mathbf{K}_{m-1}[X] \to \mathbf{K}_{m-1}[X]$ be the linear operator defined by $S(X^j) = X^{j+1}$. Then S is a simple hermitian operator of defect 1 which satisfies $V_k = (S^k x, x)$ for $k = 0, 1, \ldots, r-1$, where $x = P(X) = 1$ (see [5], pp. 348-51; [6], pp. 443-48). More generally, it was shown in [5] and [6] that the linear moment problem $V_k = (S^k x, x)$ for $k = 0, 1, \ldots, r-1$ has a solution $S \in \Lambda_s(E)$ on $x \neq 0$ if and only if the Hankel form $\mathcal{H}_{[\frac{r+1}{2}]}^A$ is positive semidefinite and $rk\,\mathcal{H}_p^A = \min(p, m)$ for $p = 1, 2, \ldots, [\frac{r+1}{2}]$, where $rk\,\mathcal{H}_p^A$ is the rank of $\mathcal{H}_p^A$ and $[z]$ is the integer defined by $[z] \leq z < [z]+1$ for $z \in \mathbf{R}$. Let Ω_r be the set of $A = (V_0, \ldots, V_{r-1}) \in \mathbf{R}^r$ such that $\mathcal{H}_{[\frac{r+1}{2}]}^A$ is positive semidefinite and $rk\,\mathcal{H}_p^A = \min(p, m)$ for $p = 1, \ldots, [\frac{r+1}{2}]$. Then, for a sequence (1), we derive the following result from Theorem 2.1.

Theorem 3.1: Let $\{V_n\}_{n\geq 0}$ be a sequence (1) and set $A = (V_0, \ldots, V_{r-1})$. Then the following statements are equivalent:

(i) $A = (V_0, \ldots, V_{r-1}) \in \Omega_r$.

(ii) The reduced linear moment problem (3), $V_n = (S^n x, x)$ for $0 \leq n \leq r-1$, has a solution $S \in \Lambda_s^P(E)$ on a nonvanishing vector $x \in E$.

(iii) The linear moment problem (2), $V_n = (S^n x, x)$ for $n \geq 0$, has a solution $S \in \Lambda_s^P(E)$ on a nonvanishing vector $x \in E$.

In (ii) and (iii), we have $x \neq 0$ and $x \in L$, an S-invariant subspace of E.

3.3 The Case of Fibonacci and Lucas Numbers

Let $(E, (.,.))$ be a unitary real vector space of dimension 2. Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 2$. Then the associated Hankel matrix

$$H = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

is positive semidefinite and has rank 2. Thus, the reduced linear moment problem,

$$L_0 = 2 = (x, x),\ L_1 = 1 = (Sx, x),\ L_2 = 3 = (S^2x, x),$$

is solvable. Then Theorem 2.1 implies that the linear moment problem is solvable for all $L_n = (S^n x, x)$, and because

$$\sigma_P = \left\{\phi_+ = \frac{1+\sqrt{5}}{2},\ \phi_- = \frac{1-\sqrt{5}}{2}\right\},$$

we derive from the method of construction of S (see Subsection 2.3) that we can choose

$$S = \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix} \quad \text{and} \quad x = (1, 1).$$

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Then the associated Hankel matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is not positive semidefinite. Therefore, even the reduced linear moment problem $F_0 = 0 = (x, x)$, $F_1 = 1 = (Sx, x)$, and $F_2 = 1 = (S^2x, x)$ is not solvable.

Because of Theorem 2.1 and Proposition 2.2, we can consider $(E, (.,.))$ as a unitary real vector space of dimension $m \geq 2$.

4. DEFINITE AND INDEFINITE LINEAR MOMENT PROBLEM FOR SEQUENCES (1)

Let $(E, (.,.))$ be a unitary real vector space of finite dimension m, and consider a sequence of real numbers $(V_n)_{0\leq n\leq p}$, where $p \leq \infty$. The linear moment problem (2) is called *definite* if it has a unique (up to conjugation by a unitary operator) solution S and *indefinite* if not. It was shown in [5] and [6] that the linear moment problem (2) is definite if and only if $p \geq 2m-1$. Let $\{V_n\}_{n\geq 0}$ be a sequence (1). Theorem 2.1 shows that if the linear moment problem (2) is solvable, it is reduced to the linear moment problem (3), $V_n = (S^n x, x)$ for $n = 0, 1, \ldots, r-1$.

Suppose that the reduced linear moment problem (3) has a solution $S \in \Lambda_s(E)$. Then the Hankel form $\mathcal{H}^A_{[\frac{r+1}{2}]}$ is positive semidefinite, and $rk\mathcal{H}^A_p = \min(p, m)$ for $p = 1, 2, \ldots, [\frac{r+1}{2}]$ (see [5], [6]), and from Theorem 2.1, we also have $S \in \Lambda^P_s(E)$. Therefore, in this case, the definite or indefinite reduced linear moment problem (3) depends on the cardinality l of the set $\sigma(S) \cap \sigma_P = \{\lambda_1, \ldots, \lambda_l\}$. Even more precisely, let $S \in \Lambda^P_s(E)$ and $\{e_1, e_2, \ldots, e_l\}$ be an orthogonal basis of $L = \oplus^l_{j=1} E_j$, where $e_j \in E_j$. Then the scalars α_j of the vector $x = \sum^l_{j=1} \alpha_j e_j$ in the reduced linear moment problem (3) satisfy the following linear system of r equations,

$$\lambda_1^j y_1 + \cdots + \lambda_l^j y_l = V_j,\ j = 0, \ldots, r-1,$$

with $y_j = \alpha_j^2$. Using Propositions 2.1 and 2.2 and Theorem 3.1, we derive the following result.

Theorem 4.1: Let $S \in \Lambda_s(E)$ and $(V_n)_{n\geq 0}$ be a sequence (1) with $r \geq 3$. Suppose $\sigma(S) \cap \sigma_P = \{\lambda_1, \ldots, \lambda_t\}$, where $l \geq 2$. Then, if the linear moment problem (3) has a solution, it is definite if $l = m \leq [\frac{r+1}{2}]$ and indefinite if $l \leq [\frac{r+1}{2}] < m$.

Example 4.1: Let $(V_n)_{n\geq 0}$ be a sequence (1) with $r\geq 3$. Suppose that the Hankel form $\mathcal{H}^A_{[\frac{r+1}{2}]}$ is positive semidefinite and $rk\,\mathcal{H}^A_p = \min(p, m)$ for $p = 1, 2, \ldots, [\frac{r+1}{2}]$. Then, by Theorem 3.1, the reduced moment problem (3) has a solution $S\in\Lambda_s(E)$. If $\sigma(S)\cap\sigma_P = \{\lambda_1, \lambda_2\}$ with $\lambda_1 < \lambda_2$, we set $L = E_1\oplus E_2$. Then $P(S_{|L}) = 0$, and we have $V_n = (S^n x, x) = \lambda_1^n a^2 + \lambda_2^n b^2$ for $n = 0, 1$, where $x = ae_1 + be_2$. For $m = 2$, Theorem 4.1 shows that the operator S is unique and that the linear moment problem (2) for the sequence (1) is definite. If $3\leq m\leq [\frac{r+1}{2}]$, Proposition 2.1 implies that, for any self-adjoint extension S_1 of $S_{|L}$ such that $\sigma(S_1)\cap\sigma_P = \{\lambda_1, \lambda_2\}$, we also have that $V_n = (S_1^n x, x)$ for $n\geq 0$. Thus, the operator S is not a unique solution of the reduced linear moment problem (2). Hence, the linear moment problem (2) for the sequence (1) is indefinite. □

5. APPLICATIONS AND CONCLUDING REMARKS

5.1 Application 1: Sequences (1) Which Are Linear Combinations of Geometric Sequences

Let $\{V_n\}_{n\geq 0}$ be a sequence (1). It is well known that $V_n = \sum_{l=1}^{k}\sum_{j=0}^{m_l-1}\beta_{l,j}\lambda_l^n$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are roots of the characteristic polynomial of the sequence (1), with multiplicities $m_1, m_2, \ldots, m_k$, respectively ($m_1 + m_2 + \cdots + m_k = r$), and $\beta_{l,j}$ are obtained from the initial conditions ($V_0, V_1, \ldots, V_{r-1}$) (see [7] and [8]). Then $\{V_n\}_{n\geq 0}$ is a linear combination of real geometric sequences if and only if

$$\beta_{l,j} = 0;\; j = 1, \ldots, m_l - 1,\; l = 1, \ldots, k, \tag{4}$$

and $\beta_{l,0} = 0$ if λ_l is a complex root. The choice of the initial conditions $(V_0, V_1, \ldots, V_{r-1})$ such that the $\beta_{l,j}$ satisfies the system of equations (4) implies that $V_n = \sum_{l=1}^{k}\beta_{l,0}\lambda_l^n$, with $\beta_{l,0} = 0$ if λ_l is a complex root. It seems difficult to find such $(V_0, V_1, \ldots, V_{r-1})$ by a direct computation from the system of equations (4). Meanwhile, Theorems 2.1 and 3.1 allow us to answer this question, as was shown in Examples 2.1-2.3.

5.2 Application 2: Sequences (1) and Linear Systems of Vandermonde Type

Consider the linear system of r equations and m unknowns $y_1, \ldots, y_m$ of Vandermonde type

$$\lambda_1^j y_1 + \cdots + \lambda_m^j y_m = b_j,\; j = 0, 1, \ldots, r-1, \tag{5}$$

where $r\geq m$ and $\lambda_j\in\mathbf{R}$ with $\lambda_i\neq\lambda_j$ if $i\neq j$. The preceding results may be used to study this system. More precisely, we can associate to this linear system of equations a sequence (1) such that $(V_0, \ldots, V_{r-1}) = (b_0, \ldots, b_{r-1})$ and whose coefficients $a_0, \ldots, a_{r-1}$ are given by the characteristic polynomial $P(X) = (X-\lambda_1)\cdots(X-\lambda_m)Q(X)$, where $Q(X)$ is a polynomial of degree $r-m$. We now consider the linear moment problem (2) for $\{V_n\}_{n\geq 0}$ with $S\in\Lambda_s(E)$, where $(E, (.,.))$ is a unitary real vector space of dimension m such that $\sigma(S) = \{\lambda_1, \ldots, \lambda_m\}$. Hence, if $\{V_n\}_{n\geq 0}$ is a sequence of moments of an operator $S\in\Lambda_s(E)$, the linear system (5) has a solution $(y_1, \ldots, y_m)$ with $y_j\geq 0$. Conversely, suppose that the system (5) has a solution $(y_1, \ldots, y_m)$ with $y_j\geq 0$. Let $(E, (.,.))$ be a unitary real vector space of dimension m and set $S\in\Lambda_s(E)$ such that $\sigma(S) = \{\lambda_1, \ldots, \lambda_m\}$. Let $\{e_1, e_2, \ldots, e_m\}$ be an orthogonal basis of E, where $Se_j = \lambda_j e_j$. Then we can verify that $V_n = (S^n x, x)$ for all $n\geq 0$, where $x = \sum_{j=1}^{m}\mu_j e_j$ with $\mu_j^2 = y_j$.

5.3 Relation with Scalar Spectral Measures

Let $(E, (.,.))$ be a unitary real vector space of dimension m and set $S \in \Lambda_s(E)$ such that $\sigma(S) = \{\lambda_1, \dots, \lambda_m\}$. Consider the spectral decomposition $E = \oplus_{j=1}^{m} E_j$ of E. For all $x = \sum_{j=1}^{m} x_j$ and $y = \sum_{j=1}^{m} y_j$ in E, where x_j and y_j are in E_j $(1 \le j \le m)$, the scalar spectral measure $\nu_{x,y}$ is defined by

$$\int_{\sigma(S)} f(t)\, d\nu_{x,y}(t) = (f(S)x, y), \tag{6}$$

where f is a continuous function on $\sigma(S)$, which may be identified with a finite sequence $(\alpha_1, \dots, \alpha_m)$. From expression (6), we derive $\nu_{x,y} = \sum_{j=1}^{m} \nu_{x_i, y_j}$, and it is easy to see that

$$\int_{\sigma(S)} f(t)\, d\nu_{x_i, y_i}(t) = (f(S)x_i, y_i) = f(\lambda_i)(x_i, y_i).$$

Thus, $\nu_{x_i, y_i} = (x_i, y_i)\delta_{\lambda_i}$, where δ_λ is the Dirac measure. In particular, for $f(z) = z^n$, we have $(f(S)x_i, y_i) = (x_i, y_i)\lambda_i^n$. Let $\{V_n\}_{n \ge 0}$ be a sequence (1) and suppose that it is a sequence of moments of the operator S on a vector $x = \sum_{j=1}^{l} \mu_j e_j$. Then we have $(x_j, y_j) = \mu_j^2$, μ_j^2 satisfies the linear system of equations of Vandermonde type (5), and $\{V_n\}_{n \ge 0}$ is a sequence of moments of the positive measure $\nu_{x,x} = \sum_{j=1}^{m} \mu_j^2 \delta_{\lambda_i}$ on $\sigma(S)$. This measure is unique if the moment problem (2) (or (3)) is definite.

In general, we can consider the measure moment problem for sequences (1) on the interval [0, 1]; it can be formulated as follows: Characterize sequences (1) that are sequences of moments ${}_n = \int_0^1 t^n d\nu(t)$ of a (unique) positive Borel measure ν on [0, 1] (see, e.g., [1]-[4]). We have found some results on this question using techniques presented in [1]-[4].

5.4 Complex Case

Suppose that $(E, (.,.))$ is a unitary complex vector space of finite dimension m. Then all results still hold.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referee for many useful remarks, observations, and suggestions that improved this paper. We also thank Professors G. Cassier, R. Curto, O. El Fallah, and E. H. Zerouali for their helpful discussions and encouragement.

REFERENCES

1. G. Cassier. "Problème des moments n-dimensionnel, mesures quasi-spectrales et semi-groupes." *Thèse de Troisième Cycle, Université C. Bernard-Lyon* **1** (1983) [or *Publ. Dép. Math.* (NS)-Lyon (1983)].
2. R. Curto & L. Fialkow. "Recursiveness, Positivity, and Truncated Moment Problems." *Houston J. Math.* **17.4** (1991):603-35.
3. R. Curto & L. Fialkow. "Solution of the Truncated Complex Moment Problem for Flat Data." *Mem. Amer. Math. Soc.* **119.568** (1996).
4. L. Fialkow. "Positivity, Extensions and the Truncated Complex Moment Problem." *Multivariable Operator Theory, Contemporary Mathematics* **185:**133-50. Providence, RI: Amer. Math. Soc., 1995.

5. I. Glazman & Y. Liubitch. *Analyse linéaire dans les espaces de dimensions finies.* 2nd ed. Moscow: Editions MIR, 1974.
6. I. M. Glazman & Ju. I. Ljubic. *Finite-Dimensional Linear Analysis: A Systematic Presentation in Problem Form.* Tr. and Ed. by G. P. Barker & G. Kuerti. Cambridge, MA: MIT Press, 1974.
7. J. A. Jeske. "Linear Recurrence Relations, Part I." *The Fibonacci Quarterly* **1.1** (1963):68-74.
8. W. G. Kelly & A. C. Peterson. *Difference Equations: An Introduction with Applications.* San Diego, CA: Academic Press, 1991.
9. E. P. Miles. "Generalized Fibonacci Numbers and Associated Matrices." *American Math. Monthly* **67** (1960):745-52.

AMS Classification Numbers: 40A05, 40A25, 47B

❖❖❖

IN MEMORIAM

Herta Taussig Freitag

December 1908-January 2000

Herta Taussig Freitag, long-time teacher, mathematician, and Fibonacci enthusiast, died January 25 at the age of 91. Her radiant smile and articulate speech reflecting her native Austria were unforgettable to hundreds of colleagues and friends. She remained an active participant in The Fibonacci Association until shortly before her death, assisting in the presentation of four papers at its 8th International Conference in 1998.

Herta's life story is one of triumph over adversity. Born in Vienna, she pursued her education there with a major interest in mathematics. When Hitler took over Austria in 1938, an event for which she had vivid memories, she began a six-year struggle to emigrate to the United States. It became clear that the only way out of Nazi Austria without a financial guarantor was to obtain employment in England as a domestic servant, an experience her brother describes as "Dickensian." A more complete account of her journey to freedom is included in the book *One-Way Ticket* by former student Mary Ann Johnson.

Upon arrival in the United States in 1944, Herta first taught at Greer School in New York State where she met her husband, Arthur H. Freitag. She began her long career at Hollins College in 1948 and completed her Ph.D. at Columbia in 1953. Among her numerous teaching awards were the Hollins Medal and the Virginia College Mathematics Teacher of the Year Award.

In her lifetime, Herta experienced prejudice in several forms but was never embittered by it. When she received the Humanitarian Award from the National Conference of Christians and Jews in 1997, the nomination read, in part: "What would have been a life-shattering experience for many set her on a course of personal professional achievement directed toward helping everyone, regardless of race, sex, color, ethnic background, religious persuasion or social class reach their maximum potential. And she does it in such a way as to make one feel that she is traveling with you, rather than leading the way."

Herta is survived by a brother, Walter Taussig, an associate conductor with the Metropolitan Opera, and a niece, Lynn Taussig. She will also be greatly missed by her Fibonacci "family" and a host of friends.

Margie Ribble

❖❖❖