THE INTEGRITY OF SOME INFINITE SERIES

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1. INTRODUCTION

Consider a sequence $\{W_n\}$ defined by the recurrence relation

$$W_n = pW_{n-1} - qW_{n-2},\ n \geq 2,\ W_0 = a,\ W_1 = b,$$

where a, b, p, and q are integers with $p > 0$, $q \neq 0$, and $\Delta = p^2 - 4q > 0$. We are interested in the following two special cases of $\{W_n\}$: $\{U_n\}$ is defined by $U_0 = 0$, $U_1 = 1$, and $\{V_n\}$ is defined by $V_0 = 2$, $V_1 = p$. It is well known that $\{U_n\}$ and $\{V_n\}$ can be expressed in the form

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ V_n = \alpha^n + \beta^n, \tag{1.1}$$

where $\alpha = (p + \sqrt{\Delta})/2$, $\beta = (p - \sqrt{\Delta})/2$. Especially, if $p = -q = 1$, $\{U_n\}$ and $\{V_n\}$ are the usual Fibonacci and Lucas sequences.

Recently, André-Jeannin studied the infinite sum (see [1])

$$S_V(x) = S_V(x; p, q) = \sum_{n=0}^{\infty} \frac{V_n}{x^n}.$$

At the end of his paper, he presented a problem to study the integrity of an infinite sum

$$T_k(x) = \sum_{n=0}^{\infty} \frac{V_{kn}}{x^n},\ k > 0.$$

In this paper, we solve this problem completely for the case in which $k \geq 2$ and $q = \pm 1$.

By using the Binet forms (1.1) and the geometric series formula, we have

$$T_k(x) = \frac{x(2x - V_k)}{x^2 - V_k x + q^k},\ |x| > \alpha^k.$$

In what follows, we shall make use of the identities:

$$U_{n+2k} - q^k U_n = U_k V_{n+k}, \tag{1.2}$$

$$U_{n+2k} + q^k U_n = V_k U_{n+k}, \tag{1.3}$$

$$V_{2n} + 2q^n = V_n^2, \tag{1.4}$$

$$V_{2n} - 2q^n = \Delta U_n^2, \tag{1.5}$$

$$U_{2n} = U_n V_n, \tag{1.6}$$

$$V_k V_{2k(n+1)} + \Delta U_k U_{2k(n+1)} = 2V_{k(2n+3)}. \tag{1.7}$$

All the identities can be obtained by the Binet form (1.1).

2. MAIN RESULTS

Let

$$V_n' = V_{kn} = \alpha^{kn} + \beta^{kn},\ U_n' = \frac{\alpha^{kn} - \beta^{kn}}{\alpha^k - \beta^k} = \frac{U_{kn}}{U_k}. \tag{2.1}$$

In fact, the sequences $\{U_n'\}$ and $\{V_n'\}$ satisfy the recurrence relation $W_n = V_k W_{n-1} - q^k W_{n-2}$. From (2.1) and applying Theorems 2-4 in [1] to the sequence $\{V_n'\}$, we can obtain the main results of this paper.

Theorem 1: If $q = \pm 1$ and $k \geq 2$, there do not exist negative rational values such that $T_k(x)$ is an integer.

Theorem 1 is a direct consequence of Theorem 2 in [1], and its proof is omitted.

Theorem 2: If $q = -1$ and $r \geq 0$, the positive rational values of x for which $T_{2r+1}(x)$ is integral are given by

$$x = \frac{U_{(2r+1)(2m+1)}}{U_{2m(2r+1)}} \quad (m = 1, 2, \ldots)$$

and

$$x = \frac{V_{(2r+1)(2m+2)}}{V_{(2r+1)(2m+1)}} \quad (m = 0, 1, 2, \ldots).$$

The corresponding values of $T_{2r+1}(x)$ are given by

$$T_{2r+1}\left(\frac{U_{(2r+1)(2m+1)}}{U_{2m(2r+1)}}\right) = \frac{U_{(2r+1)(2m+1)}V_{2m(2r+1)}}{U_{2r+1}}$$

and

$$T_{2r+1}\left(\frac{V_{(2r+1)(2m+2)}}{V_{(2r+1)(2m+1)}}\right) = \frac{U_{(2r+1)(2m+1)}V_{(2r+1)(2m+2)}}{U_{2r+1}}.$$

Proof: Since $q = -1$, we can apply Theorem 3 in [1] to the sequence $\{V_n' = V_{(2r+1)n}\}$. Therefore, the positive rational values of x for which $T_{2r+1}(x)$ is integral are given by

$$x = \frac{U_{2m+1}'}{U_{2m}'} \quad (m = 1, 2, \ldots)$$

and

$$x = \frac{V_{2m+2}'}{V_{2m+1}'} \quad (m = 0, 1, 2, \ldots).$$

The corresponding values of $T_{2r+1}(x)$ are given by

$$T_{2r+1}\left(\frac{U_{2m+1}'}{U_{2m}'}\right) = U_{2m+1}'V_{2m}'$$

and

$$T_{2r+1}\left(\frac{V_{2m+2}'}{V_{2m+1}'}\right) = U_{2m+1}'V_{2m+2}'.$$

From (2.1), we can obtain the results. □

Theorem 3: If $q=1$, $p\geq 3$, and $r\geq 0$, the positive rational values of x for which $T_{2r+1}(x)$ is integral are given by

$$x=\frac{U_{(2r+1)(m+1)}}{U_{m(2r+1)}}\quad(m=1,2,\ldots)$$

and

$$x=\frac{X_{(2r+1)(m+1)}}{X_{m(2r+1)}}\quad(m=0,1,2,\ldots),$$

where $X_{m(2r+1)}=U_{(2r+1)(m+1)}+U_{m(2r+1)}$. The corresponding values of $T_{2r+1}(x)$ are given by

$$T_{2r+1}\left(\frac{U_{(2r+1)(m+1)}}{U_{m(2r+1)}}\right)=\frac{U_{(2r+1)(m+1)}V_{m(2r+1)}}{U_{2r+1}}$$

and

$$T_{2r+1}\left(\frac{X_{(2r+1)(m+1)}}{X_{m(2r+1)}}\right)=\frac{(U_{(2r+1)(m+1)}-U_{m(2r+1)})X_{(2r+1)(m+1)}}{U_{2r+1}^2}.$$

Proof: By $q=1$ and $p\geq 3$, we have that

$$V_{2r+1}=\alpha^{2r+1}+\alpha^{-(2r+1)}\geq 3.$$

Similarly to the proof of the last theorem, we can prove the conclusion from Theorem 4 in [1]. □

Theorem 4: If $q=-1$ and $r\geq 1$, the positive rational values of x for which $T_{2r}(x)$ is integral are given by

$$x=\frac{U_{r(m+2)}}{U_{rm}}\quad(r\text{ or }m\text{ even and }m\geq 1)$$

and

$$x=\frac{V_{r(2m+3)}}{V_{r(2m+1)}}\quad(r\text{ odd and }m\geq 0).$$

The corresponding values of $T_{2r}(x)$ are given by

$$T_{2r}\left(\frac{U_{r(m+2)}}{U_{rm}}\right)=\frac{U_{r(m+2)}V_{rm}}{U_{2r}}$$

and

$$T_{2r}\left(\frac{V_{r(2m+3)}}{V_{r(2m+1)}}\right)=\frac{U_{r(2m+1)}V_{r(2m+3)}}{U_{2r}}.$$

Proof: Apply Theorem 4 in [1] to the sequence $\{V_n'=V_{2rn}\}$. Therefore, the positive rational values of x for which $T_{2r}(x)$ is integral are given by

$$x=\frac{U_{m+1}'}{U_m'}\quad(m=1,2,\ldots)$$

and

$$x=\frac{X_{m+1}'}{X_m'}\quad(m=0,1,\ldots),$$

where $X_m'=U_{m+1}'+U_m'$. The corresponding values of $T_{2r}(x)$ are given by

$$T_{2r}\left(\frac{U'_{m+1}}{U'_m}\right) = U'_{m+1}V'_m$$

and

$$T_{2r}\left(\frac{X'_{m+1}}{X'_m}\right) = X'_{m+1}(U'_{m+1} - U'_m).$$

From (2.1), we have

$$\frac{U'_{m+1}}{U'_m} = \frac{U_{2r(m+1)}}{U_{2rm}} \quad \text{and} \quad X'_{m+1} = \frac{U_{2r(m+2)} + U_{2r(m+1)}}{U_{2r}}.$$

It follows from (1.2) and (1.3) that

$$X'_{m+1} = \frac{U_{2r}V_{2r(m+1)} + V_{2r}U_{2r(m+1)} + 2U_{2r(m+1)}}{2U_{2r}}.$$

From (1.2)-(1.7), we have

$$X'_{m+1} = \begin{cases} \frac{U_{r(2m+3)}}{U_r} & r \text{ is even,} \\ \frac{V_{r(2m+3)}}{V_r} & r \text{ is odd.} \end{cases}$$

Using a similar method, we have

$$X'_m = \begin{cases} \frac{U_{r(2m+1)}}{U_r} & r \text{ is even,} \\ \frac{V_{r(2m+3)}}{V_r} & r \text{ is odd.} \end{cases}$$

It follows from (1.2) and (1.6) that

$$U'_{m+1} - U'_m = \frac{U_{2r(m+1)} - U_{2rm}}{U_{2r}} = \frac{U_rV_{r(2m+1)} + ((-1)^r - 1)U_{2rm}}{U_{2r}}$$

$$= \begin{cases} \frac{V_{r(2m+1)}}{V_r} & r \text{ is even,} \\ \frac{U_rV_{r(2m+1)} - 2U_{2rm}}{U_{2r}} & r \text{ is odd.} \end{cases}$$

When $q = -1$ and r is odd, from (1.2) and (1.3) we have

$$U'_{m+1} - U'_m = \begin{cases} \frac{V_{r(2m+1)}}{V_r} & r \text{ is even,} \\ \frac{U_{r(2m+1)}}{U_r} & r \text{ is odd.} \end{cases}$$

Therefore, the positive rational values of x for which $T_{2r}(x)$ is integral are given by

$$x = \frac{U_{2r(m+1)}}{U_{2rm}} \quad (m \geq 1), \tag{2.2}$$

$$x = \frac{U_{r(2m+3)}}{U_{r(2m+1)}} \quad (r \text{ even and } m \geq 0), \tag{2.3}$$

and

$$x = \frac{V_{r(2m+3)}}{V_{r(2m+1)}} \quad (r \text{ odd and } m \geq 0).$$

The corresponding values of $T_{2r}(x)$ are given by

$$T_{2r}\left(\frac{U_{2r(m+1)}}{U_{2rm}}\right)=\frac{U_{2r(m+1)}V_{2rm}}{U_{2r}}, \tag{2.4}$$

$$T_{2r}\left(\frac{U_{r(2m+3)}}{U_{r(2m+1)}}\right)=\frac{U_{r(2m+3)}V_{r(2m+1)}}{U_{2r}} \quad (r \text{ even}), \tag{2.5}$$

and

$$T_{2r}\left(\frac{V_{r(2m+3)}}{V_{r(2m+1)}}\right)=\frac{U_{r(2m+1)}V_{r(2m+3)}}{U_{2r}} \quad (r \text{ odd}).$$

It is clear that (2.2) and (2.3) can be rewritten as

$$x=\frac{U_{r(m+2)}}{U_{rm}}.$$

Similarly, (2.4) and (2.5) can be rewritten as

$$T_{2r}\left(\frac{U_{r(m+2)}}{U_{rm}}\right)=\frac{U_{r(m+2)}V_{rm}}{U_{2r}}.$$

On the other hand, since $q=-1$, $\frac{U_{r(m+2)}}{U_{rm}}>\alpha^{2r}$ holds when m or r is even. Hence, the conclusions are valid. □

Theorem 5: If $q=1$, $p\geq 3$, and $r\geq 1$, the positive rational values of x for which $T_{2r}(x)$ is integral are given by

$$x=\frac{U_{r(m+2)}}{U_{rm}} \quad (m=1,2,\ldots),$$

and the corresponding values of $T_{2r}(x)$ are given by

$$T_{2r}\left(\frac{U_{r(m+2)}}{U_{rm}}\right)=\frac{U_{r(m+2)}V_{rm}}{U_{2r}}.$$

Proof: Since $q=1$ and $V_{2r}\geq 3$, we can apply Theorem 4 in [1] to the sequence $\{V_n'=V_{2rn}\}$. The proof is similar to the one of the last theorem. □

Clearly, André-Jeannin's results are special cases of Theorems 2 and 3.

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REFERENCE

1. R. André-Jeannin. "On the Integrity of Certain Infinite Series." *The Fibonacci Quarterly* **36.2** (1998):174-80.

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