ON r-GENERALIZED FIBONACCI SEQUENCES AND HAUSDORFF MOMENT PROBLEMS

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1. INTRODUCTION

Let $a_0, a_1, \ldots, a_{r-1}$ with $a_{r-1} \neq 0$ $(r \geq 2)$ be fixed complex numbers. For any sequence of complex numbers $A = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})$, we define the r-generalized Fibonacci sequence $\{Y_A(n)\}_{n\geq 0}$ as follows: $Y_A(n) = \alpha_n$ for $n = 0, 1, \ldots, r-1$ and

$$Y_A(n+1) = a_0 Y_A(n) + a_1 Y_A(n-1) + \cdots + a_{r-1} Y_A(n-r+1) \tag{1}$$

for all $n \geq r-1$. Such sequences have been studied in the literature (see, e.g., [5], [6], and [8]-[12]).

Let $\gamma = \{\gamma_n\}_{0\leq n\leq p}$, where $p \leq +\infty$, be a sequence of real numbers. The Hausdorff moment problem associated with γ consists of finding a positive Borel measure μ such that

$$\gamma_n = \int_a^b t^n d\mu(t) \text{ for all } n \ (0 \leq n \leq p) \text{ and } Supp(\mu) \subset [a, b], \tag{2}$$

where $Supp(\mu)$ is the support of μ. If this problem has a solution μ, we say that μ is the *representing measure* of $\gamma = \{\gamma_n\}_{0\leq n\leq p}$. For $p = +\infty$, problem (2) is called the *full Hausdorff moment problem* (see, e.g., [1] and [2]). When $p < +\infty$, problem (2) is called the *truncated Hausdorff moment problem*, and it has been studied by Curto-Fialkow in [3], [4], and [7].

The aim of this paper is to study the Hausdorff moment problem on $[a, b]$ associated with an r-generalized Fibonacci sequence $\gamma = \{Y_A(n)\}_{n\geq 0}$. Some necessary and sufficient conditions for the existence of a positive Borel measure μ satisfying (2) are derived from those established for the full or truncated Hausdorff moment problem (see [1]-[4] and [7]).

This paper is organized as follows: In Section 2 we study the connection between the discrete positive measure and sequences (1). We also give two fundamental lemmas on representing measures of sequences (1). Section 3 deals with the full Hausdorff moment problem for sequences (1) using Cassier's method (see [2]). In Section 4 the Hausdorff moment problem for sequences (1) is studied using Curto-Fialkow's method (see [3]). Section 5 concerns the extension property of the truncated Hausdorff moment sequence to sequences (1).

2. SEQUENCES (1) AND REPRESENTING MEASURES

2.1. Discrete Positive Measure and Sequences (1)

Let $[a, b]$ be an interval of **R** and consider the following discrete positive measure

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{x_j},$$

where $\rho_j \in \mathbf{R}$ and $Supp(\mu) \subset [a, b]$. Let $\{\alpha_n\}_{n\geq 0}$ be the sequence of moments of μ. Hence,

$$\alpha_n = \int_a^b t^n d\mu(t) = \sum_{j=0}^{r-1} \rho_j x_j^n \quad \text{for all } n \geq 0.$$

Consider the polynomial $P_\mu(X) = \prod_{j=0}^{r-1}(X - x_j) = X^r - a_0 X^{r-1} - a_1 X^{r-2} - \cdots - a_{r-1}$. It is clear that $x_0, x_1, \ldots, x_{r-1}$ are simple roots of $P_\mu(X)$. Thus, we have $x_j^{n+1} = a_0 x_j^n + a_1 x_j^{n-1} + \cdots + a_{r-1} x_j^{n-r+1}$ $(0 \leq j \leq r-1)$ for any $n \geq r-1$. This implies that

$$\alpha_{n+1} = a_0 \alpha_n + a_1 \alpha_{n-1} + \cdots + a_{r-1} \alpha_{n-r+1} \quad \text{for all } n \geq r-1.$$

Then the moment sequence $\{\alpha_n\}_{n\geq 0}$ of $\mu = \sum_{j=0}^{r-1} \rho_j \delta_{x_j}$ is sequence (1) with coefficients $a_0, \ldots, a_{r-1}$ and initial conditions $A = (\alpha_0, \ldots, \alpha_{r-1})$.

We can see then that problem (2) for sequences (1) is nothing more than the converse of the preceding assertions.

2.2. Two Fundamental Lemmas on Representing Measures of Sequences (1)

Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1) and suppose that μ is a representing measure of $\{Y_A(n)\}_{0\leq n\leq 2r}$. Then, for any n $(0 \leq n \leq r)$, we have

$$Y_A(n+r) = \int_a^b t^{n+r} d\mu(t) = \int_a^b t^n [a_0 t^{r-1} + a_1 t^{r-2} + \cdots + a_{r-1}] d\mu(t).$$

Thus, we have $\int_a^b t^n P(t) d\mu(t) = 0$ for all n $(0 \leq n \leq r)$, where $P(X)$ is the characteristic polynomial of sequence (1). The preceding relation implies that $\int_a^b P(t)^2 d\mu(t) = 0$. Since μ is a positive Borel measure, it follows that $Supp(\mu) \subset Z(P) = \{x \in [a, b];\ P(x) = 0\}$. Hence, we have the following lemma.

Lemma 2.1: Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1). Suppose that μ is a representing measure of $\{Y_A(n)\}_{0\leq n\leq 2r}$. Then $Supp(\mu) \subset Z(P) = \{x \in [a, b];\ P(x) = 0\}$, where P is the characteristic polynomial of $\{Y_A(n)\}_{n\geq 0}$.

We note that the proof of Lemma 2.1 is identical to the proof of Lemma 3.6 of [3], but in our case $P(X)$ is the characteristic polynomial of sequence (1). It follows from Lemma 2.1 that, if sequence (1) is a moment sequence of a positive Borel measure μ on $[a, b]$, then μ is a discrete measure with $Supp(\mu) \subset Z(P)$.

Using Lemma 2.1, we can prove the following property.

Lemma 2.2 (Lemma of Reduction): Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1) and let $P(X)$ be its characteristic polynomial. Let μ be a Borel measure on $[a, b]$. Then the following statements are equivalent.

(i) μ is a representing measure of $\{Y_A(n)\}_{n\geq 0}$ on $[a, b]$.

(ii) μ is a representing measure of $\{Y_A(n)\}_{0\leq n\leq 2r}$ on $[a, b]$.

(iii) μ is a representing measure of $A = (\alpha_0, \ldots, \alpha_{r-1})$ with $Supp(\mu) \subset Z(P) = \{x \in [a, b];\ P(x) = 0\}$.

Proof: It is easy to see that $(i) \Rightarrow (ii)$. From Lemma 2.1, we derive that $(ii) \Rightarrow (iii)$. If $\alpha_j = \int_a^b t^j d\mu(t)$ for $0 \le j \le r-1$ and $Supp(\mu) \subset Z(P)$, then

$$Y_A(r) = \int_a^b [a_0 t^{r-1} + a_1 t^{r-2} + \cdots + a_{r-1}] d\mu(t) = \int_a^b t^r d\mu(t).$$

By induction we have $Y_A(n) = \int_a^b t^n d\mu(t)$ for any $n \ge r$. Consequently, μ is a representing measure of $\{Y_A(n)\}_{n\ge 0}$ on $[a, b]$. $\square$

Lemma 2.2 has two important consequences. First, we can use it to see that the full Hausdorff moment problem for sequences (1) may be reduced to the truncated Hausdorff moment problem studied in [3]. Second, we shall also see that the truncated Hausdorff moment problem for a sequence $\gamma = \{\gamma_j\}_{0\le j\le n}$ can be extended to sequence (1).

3. SEQUENCES (1) AND FULL HAUSDORFF MOMENT PROBLEM

Let $\{Y_A(n)\}_{n\ge 0}$ be sequence (1) in $[0, 1]$ and $\mathbf{R}[X]$ the $\mathbf{R}$-vector space of polynomials. Consider the linear functional $L: \mathbf{R}[X] \to \mathbf{R}$ defined by $L(X^n) = Y_A(n)$ for $n \ge 0$. From relation (1), we derive that $L(X^k P(X)) = 0$ for all $k \ge 0$, which implies that $L(QP) = 0$ for any Q in $\mathbf{R}[X]$. Hence, $I = (P)$ is an ideal of $\mathbf{R}[X]$ with $(P) \subset \ker L$. Conversely, let $\{V_n\}_{n\ge 0}$ be a sequence of real numbers and $L: \mathbf{R}[X] \to \mathbf{R}$ a linear functional defined by $L(X^n) = V_n$. If there exists $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ such that $L(X^k P(X)) = 0$ for all $k \ge 0$, then $\{V_n\}_{n\ge 0}$ is given by sequence (1) with coefficients $a_0, \ldots, a_{r-1}$ and initial conditions $A = (V_0, \ldots, V_{r-1})$.

Proposition 3.1: Let $\{V_n\}_{n\ge 0}$ be a sequence of real numbers and $L: \mathbf{R}[X] \to \mathbf{R}$ a linear functional defined by $L(X^n) = V_n$ for $n \ge 0$. Then:

(i) If $\{V_n\}_{n\ge 0}$ is given by sequence (1) with characteristic polynomial P, we have $I = (P) \subset \ker L$.

(ii) If there exists a polynomial $P = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ $(r \ge 2)$ such that $I = (P) \subset \ker L$, then $\{V_n\}_{n\ge 0}$ is given by sequence (1) with coefficients $a_0, \ldots, a_{r-1}$ and initial conditions $A = (V_0, \ldots, V_{r-1})$.

Let $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ $(a_{r-1} \ne 0)$ and let $R(X)$ be in $\mathbf{R}[X]$ such that $R(X) \ge 0$ for all x in $[0, 1]$. It is well known that there exists A and B in $\mathbf{R}[X]$ such that $R(X) = A(X)^2 + X(1-X)B(X)^2$ (see, e.g., [2]). Since $A = Q_1 P + A_1$ and $B = Q_2 P + B_1$, where Q_1, Q_2, A_1, and B_1 are in $\mathbf{R}[X]$, with $\deg A_1 \le r-1$ and $\deg B_1 \le r-1$, we derive the following lemma.

Lemma 3.2: Let $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ $(a_{r-1} \ne 0)$ and $R(X) \in \mathbf{R}[X]$ such that $R(x) \ge 0$ for all x in $[0, 1]$. Then there exist Q, A_1, A_2 in $\mathbf{R}[X]$ such that $R(X) = Q(X)P(X) + A_1^2 + X(1-X)B_1^2$, where $\deg A_1 \le r-1$ and $\deg B_1 \le r-1$.

We recall that a real matrix $M = [m_{ij}]_{0\le i,\, j\le k}$ $(k \le +\infty)$ is *positive* if, for all (finite) real sequences $\{\xi_j\}_{0\le j\le p}$, we have

$$\sum_{i,\, j=0}^{k} m_{ij} \xi_i \xi_j \ge 0.$$

Note that $M \ge 0$. It was proved in [2] (see Theorem 1.2.3) that a real sequence $\{Y_n\}_{n\ge 0}$ is a moment sequence of Borel positive measure μ on $[0, 1]$ if and only if the two matrices

$$M = [Y_{i+j}]_{i,\, j\geq 0} \quad \text{and} \quad N = [Y_{i+j+1} - Y_{i+j+2}]_{i,\, j\geq 0}$$

are positive. From Proposition 3.1 and Lemma 3.2, we derive the following result.

Theorem 3.3: Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1). Then $\{Y_A(n)\}_{n\geq 0}$ is a moment sequence of a unique positive Borel measure μ on [0, 1] if and only if the following two matrices,

$$H(r) = [Y_A(i+j)]_{0\leq i,\, j\leq r-1} \quad \text{and} \quad K(r) = [Y_A(i+j+1) - Y_A(i+j+2)]_{0\leq i,\, j\leq r-1}, \tag{3}$$

are positive.

Proof: Suppose that the two matrices $H(r)$ and $K(r)$ as defined in (3) are positive. Let $L: \mathbf{R}[X] \to \mathbf{R}$ be a linear functional defined by $L(X^n) = V_n$ for $n \geq 0$. For any $R \in \mathbf{R}[X]$ such that $R(x) \geq 0$ for any $x \in [0, 1]$, Lemma 3.2 implies that $R = QP + A_1^2 + X(1-X)B_1^2$, where Q, A_1, $B_1 \in R[X]$ with $\deg A_1 \leq r-1$ and $\deg B_1 \leq r-1$. If $A_1(X) = \sum_{j=0}^{r-1} \lambda_j X^j$ and $B_1(X) = \sum_{j=0}^{r-1} \beta_j X^j$, then

$$L(R) = \sum_{0\leq i,\, j\leq r-1} Y_A(i+j)\lambda_i\lambda_j + \sum_{0\leq i,\, j\leq r-1} [Y_A(i+j+1) - Y_A(i+j+2)]\beta_i\beta_j\,.$$

Since $H(r)$ and $K(r)$ are positive, we obtain $L(R) \geq 0$. Consider the Banach space

$$\left(C([0, 1], \mathbb{R}),\ \|\cdot\|_{[0,1]}\right)$$

of continuous functions on [0, 1], where $\|f\|_{[0,1]} = \sup_{x\in[0,1]}|f(x)|$. Then $|L(R)| \leq \|R\|_{[0,1]} L(1)$. This allows us to extend the linear functional L to a positive measure μ on [0, 1], where $L(f) = \int_0^1 f(t)d\mu(t)$ for any $f \in C([0, 1], \mathbb{R})$. Thus, we have $Y_A(n) = \int_0^1 t^n d\mu(t)$ for any $n \geq 0$. Conversely, if $A(X) = \sum_{j=0}^{r-1} \lambda_j X^j$, then $A(x)^2 \geq 0$ and $x(1-x)A(x)^2 \geq 0$ for any $x \in [0, 1]$. Thus,

$$\int_0^1 A(t)^2 d\mu(t) = \sum_{0\leq i,\, j\leq r-1} Y_A(i+j)\lambda_i\lambda_j \geq 0$$

and

$$\int_0^1 t(1-t)A(t)^2 d\mu(t) = \sum_{0\leq i,\, j\leq r_1} [Y_A(i+j+1) - Y_A(i+j+2)]\lambda_i\lambda_j \geq 0.$$

Therefore, the two matrices $H(r)$ and $K(r)$ are positive. □

Using an affine transformation, it was established in [2] (see Corollary 1.2.4) that a real sequence $\{Y_n\}_{n\geq 0}$ is a moment sequence of a positive Borel measure on $[a, b]$ if and only if the two matrices

$$M = [Y_{i+j}]_{i,\, j\geq 0} \quad \text{and} \quad N = [(a+b)Y_{i+j+1} - Y_{i+j+2} - abY_{i+j}]_{i,\, j\geq 0}$$

are positive. Thus, for sequences (1), we derive the following corollary from Theorem 3.3.

Corollary 3.4: Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1). Then $\{Y_A(n)\}_{n\geq 0}$ is a moment sequence of a unique positive Borel measure μ on $[a, b]$ if and only if the two matrices

$$H(r) = [Y_A(i+j)]_{0\geq i,\, j\leq r-1} \quad \text{and} \quad K(r) = [(a+b)Y_A(i+j+1) - Y_A(i+j+2) - abY_A(i+j)]_{0\leq i,\, j\leq r-1}$$

are positive.

Theorem 3.3 and Corollary 3.4 allow us to see that the full Hausdorff moment problem for sequence (1) can be reduced to the truncated Hausdorff moment problem, which is conformable with the result of Lemma 2.2.

4. SEQUENCES (1) AND TRUNCATED HAUSDORFF MOMENT PROBLEM

The Hankel matrices associated with a given real sequence $\gamma = \{\gamma_j\}_{j\geq 0}$ are defined by $H(n) = [\gamma_{i+j}]_{0\leq i,\, j\leq n}$, where $n \geq 0$. The (*Hankel*) *rank* of the Hankel matrix $A = [\gamma_{i+j}]_{0\leq i,\, j\leq k}$, where $(\gamma_0, \ldots, \gamma_{2k})$ in $\mathbf{R}^{2k+1}$, denoted by $rank(\gamma)$ is defined as follows: If A is nonsingular, $rank(\gamma) = k+1$, and if A is singular, $rank(\gamma)$ is the smallest integer i $(1 \leq i \leq k)$ such that $V_i \in span\{V_1, \ldots, V_{i-1}\}$, where $V_j = (\gamma_{j+l})_{l=0}^{k}$ is the j^{th} column of A. Thus, if A is singular, there exists a unique $(\phi_0, \ldots, \phi_{i-1})$ in $\mathbf{R}^i$ such that $V_i = \phi_{i-1}V_0 + \cdots + \phi_0 V_{i-1}$. The polynomial $g_\gamma(X) = X^i - \phi_0 X^{i-1} + \cdots + \phi_{i-1}$ is called the *generating function* of $\gamma = (\gamma_0, \ldots, \gamma_{2k})$ (see [3]).

Let $Y_A = \{Y_A(n)\}_{n\geq 0}$ be given by the sequence (1) and consider the full Hausdorff moment problem (2) for Y_A on $[a, b]$. From Lemma 2.2, this problem may be reduced to the following truncated Hausdorff moment problem: Find necessary and sufficient conditions for the existence of a positive Borel measure μ such that

$$Y_A(n) = \int_a^b t^n d\mu(t), \ (0 \leq n \leq 2r) \quad \text{and} \quad Supp(\mu) \subset [a, b].$$

The general case for the truncated Hausdorff moment problem has been studied in [3]. Consider the two Hankel matrices

$$A(r) = [Y_A(i+j)]_{0\leq i,\, j\leq r} \quad \text{and} \quad B(r) = [Y_A(i+j+1)]_{0\leq i,\, j\leq r}.$$

Since $Y_A(n+1) = \Sigma_{j=0}^{r-1} a_j Y_A(n-j)$ for $n \geq r-1$, the column vector $V(r+1, r) = (Y_A(r+1+j))_{j=0}^{r}$ is an element of the range of $A(r)$ and the (Hankel) $rank(Y_A)$ is equal to $rank(Y_A^{(r)})$, where $Y_A^{(r)} = (Y_A(0), \ldots, Y_A(2r))$. Thus, we have $s := rank(Y_A) \leq r$. Hence, for sequence (1), the preceding Lemma 2.2 and Theorem 4.3 of [3] imply that the following are equivalent.

(i) There exists a Borel positive measure μ such that $Supp(\mu) \subset [a, b]$ and $Y_A(n) = \int_a^b t^n d\mu(t)$, $0 \leq n \leq 2r$.

(ii) There exists an r-atomic representing measure μ for Y_A such that $Supp(\mu) \subset [a, b]$.

(iii) $A(r) \geq 0$ and $bA(r) \geq B(r) \geq aA(r)$.

Consequently, we have the following result.

Theorem 4.1: Let $Y_A = \{Y_A(n)\}_{n\geq 0}$ be given by sequence (1), where $A = (\alpha_0, \ldots, \alpha_{r-1})$ with $\alpha_0 > 0$ and let $s := rank(Y_A) = rank(Y_A^{(r)})$. The following statements are equivalent.

(i) There exists a Borel positive measure μ with $Supp(\mu) \subset [a, b]$ such that $Y_A(n) = \int_a^b t^n d\mu(t)$ for all $n \geq 0$.

(ii) There exists an s-atomic representing measure μ for Y_A such that $Supp(\mu) \subset [a, b]$.

(iii) $A(r) \geq 0$ and $bA(r) \geq B(r) \geq aA(r)$.

Suppose that $\{Y_A(n)\}_{n\geq 0}$ is a moment sequence of a positive Borel measure μ on $[a,b]$. Then, from Theorem 4.1, we derive that $\mu = \sum_{j=1}^{s} \rho_j \delta_{x_j}$, where $p_j \geq 0$ and $\{x_1, \ldots, x_s\} \subset [a,b] \cap Z(P)$. The real numbers ρ_j are given by the following linear system of r equations

$$x_1^j \rho_1 + x_2^j \rho_2 + \cdots + x_s^j \rho_s = \alpha_j, \; 0 \leq j \leq r-1.$$

5. FIBONACCI EXTENSION OF γ

Let $\gamma = (Y_0, \ldots, Y_m) \in \mathbf{R}^{m+1}$ with $(Y_0 > 0)$. In [3], Curto-Fialkow give necessary and sufficient conditions for the existence of a positive Borel measure μ such that

$$Y_j = \int_a^b t^j d\mu(t) \;\text{ for } j = 0, 1, \ldots, m \;\text{ and }\; Supp(\mu) \subset [a,b]. \tag{4}$$

Let $V_i = (Y_{i+j})_{0\leq j\leq k}$ $(i = 0, \ldots, k+1)$ be the i^{th} column vector of $A(k)$ and $r = rank(\gamma)$. Thus, $\{V_1, \ldots, V_{r-1}\}$ are linearly independent, and there exists $(b_0, \ldots, b_{r-1}) \in \mathbf{R}^r$ such that $V_r = b_0 V_{r-1} + \cdots + b_{r-1} V_0$. If $V(r, r-1) = (Y_{r+j})_{j=0}^{r-1}$, then we have $(b_0, \ldots, b_{r-1}) = A(r-1)^{-1} V(r, r-1)$. For $m = 2k$ or $2k+1$, Curto-Fialkow proved in [3] that there exists a positive Borel measure μ satisfying (4) and $Supp(\mu) \subset [a,b] \cap Z(P_\gamma)$, where $r = rank(\gamma)$ and P_γ is the generating function of γ (see Theorem 4.1 and 4.3 of [3]). Since $Supp(\mu) \subset Z(P_\gamma)$, we derive that

$$Y_{j+1} = b_0 Y_j + \cdots + b_{r-1} Y_{j-r+1} \;\text{ for } r-1 \leq j \leq 2k.$$

Let $\{Y_A(n)\}_{n\geq 0}$ be given by sequence (1) defined by $A = (Y_0, \ldots, Y_{r-1})$ and $Y_A(n+1) = b_0 Y_A(n) + \cdots + b_{r-1} Y_A(n-r+1)$ for $n \geq 0$. This sequence, called the *Fibonacci extension of the truncated Hausdorff moment problem of* γ, satisfies

$$Y_A(n) = \int_a^b t^n d\mu(t) \;\text{ for all } n \geq 0.$$

Proposition 5.1: Let $\gamma = (Y_0, \ldots, Y_m)$ with $Y_0 > 0$. Suppose that there exists a positive Borel measure μ which is a representing measure of γ. Then γ owns an extension $\{Y_A(n)\}_{n\geq 0}$ which is a sequence (1), where $r = rank(\gamma)$, $A = (Y_0, \ldots, Y_{r-1})$ and the coefficients $b_0, \ldots, b_{r-1}$ are given by the characteristic polynomial P_μ of μ.

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