# THE ZECKENDORF NUMBERS AND THE INVERSES OF SOME BAND MATRICES

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## 1. INTRODUCTION

For the tridiagonal matrix

$$A=\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}_{(n-1)\times(n-1)},$$

it is known (see [1]) that $A^{-1}=[a_{ij}]$ has elements

$$a_{ij}=\begin{cases} \frac{i(n-j)}{n}, & i\le j, \\ \frac{j(n-i)}{n}, & i>j. \end{cases}$$

We will find the inverse of the matrix $A_p$, $p\in\mathbb{N}$, where

$$A_p=\begin{bmatrix} 2 & -1 & & & & & & & & & \\ 0 & 2 & -1 & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & & & & & & & \\ \cdot & & \cdot & \cdot & \cdot & & & & & & \\ 0 & \cdot & \cdot & 0 & 2 & -1 & & & & & \\ -1 & 0 & \cdot & \cdot & 0 & 2 & -1 & & & & \\ & -1 & 0 & \cdot & \cdot & 0 & 2 & -1 & & & \\ & & \cdot & \cdot & & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & & & \cdot & \cdot & \cdot & \\ & & & & -1 & 0 & \cdot & \cdot & 0 & 2 & -1 \\ & & & & & -1 & \underbrace{0 & \cdot & \cdot & 0}_{p-1} & 2 \end{bmatrix}_{(n-1)\times(n-1)}.$$

It is evident that, for $p=1$, $A_p=A$.

Let $F_j^{(p)}$, $p\in\mathbb{N}$, $j\in\mathbb{N}\cup\{0\}$, be the Zeckendorf numbers given in [2] with

$$F_0^{(p)}=0,\ \ p\in\mathbb{N},$$
$$F_1^{(p)}=1,\ \ p\in\mathbb{N},$$

and

$$F_j^{(p)} = \begin{cases} 1, & p = 1, j \in \mathbb{N}, \\ 2^{j-2}, & p \geq 2, j \in \{2, 3, \dots, p\}, \\ F_{j-1}^{(p)} + F_{j-2}^{(p)} + \cdots + F_{j-p}^{(p)}, & p \geq 2, j > p, \end{cases}$$

and let us define

$$\begin{aligned} u_i^{(0)} &= \sum_{j=1}^{i} F_j^{(p)}, \quad i \in \{1, 2, \dots, n\}, \\ u_i^{(p)} &= 0, \quad i \leq 0. \end{aligned} \tag{1}$$

***Theorem (Main Result):*** If the numbers $u_i^{(p)}$ are as in (1), define the matrix $A'_p = [a_{ij}]$, $p \geq 2$, by

$$a_{ij} = \frac{1}{u_n^{(p)}} \left( u_i^{(p)} u_{n-j}^{(p)} - u_n^{(p)} u_{i-j}^{(p)} \right), \quad i, j \in \{1, 2, \dots, n-1\}. \tag{2}$$

Then $A'_p = A_p^{-1}$.

It is important to mention that, since $p \geq 2$, $n$ must not be less than 4.

## 2. PROOF OF THE MAIN RESULT

First, we will establish some properties of the numbers $u_i^{(p)}$.

***Lemma 1:*** For $i, p \in \mathbb{N}$, $p \geq 2$, and $i \leq p+1$,

$$u_i^{(p)} = 2^{i-1}.$$

***Proof:*** If $i \leq p$, then

$$u_i^{(p)} = \sum_{j=1}^{i} F_j^{(p)} = 1 + (1 + 2 + \cdots + 2^{i-2}) = 1 + \frac{2^{i-1} - 1}{2 - 1} = 2^{i-1},$$

and if $i = p+1$, then

$$u_{p+1}^{(p)} = u_p^{(p)} + F_{p+1}^{(p)} = 2\sum_{j=1}^{p} F_j^{(p)} = 2^p. \quad \square$$

***Lemma 2:*** If $\delta_{kl}$ is the Kronecker delta symbol, then

$$u_{k-p-l}^{(p)} - 2u_{k-l}^{(p)} + u_{k+1-l}^{(p)} = \delta_{kl}, \tag{3}$$

for $p, k \in \mathbb{N} \setminus \{1\}$ and $l \in \mathbb{N} \cup \{0\}$.

***Proof:*** Let us consider two different cases: ***(a)*** $l \geq k$; ***(b)*** $l < k$.

***(a)*** For $l \geq k$m we have $k - p - l < 0$, $k - l \leq 0$, and $k + 1 - l \leq 1$. Hence, by (1),

$$u_{k-p-l}^{(p)} = u_{k-l}^{(p)} = 0,$$

$$u_{k+1-l}^{(p)} = \begin{cases} 0, & k < l, \\ F_1^{(p)} = 1, & k = l, \end{cases}$$

and (3) is valid.

***(b)*** For $l<k$, first let $2 \le k \le p$. Then $k-p-l \le 0$ and $k-l<k+1-l \le p+1$. Hence, by (1) and Lemma 1,

$$u_{k-p-l}^{(p)}=0,\ u_{k-l}^{(p)}=2^{k-l-1}, \text{ and } u_{k+1-l}^{(p)}=2^{k-l}.$$

It follows that (3) is true.

If $k \ge p+1$, then for: ***(i)*** $0<k-l \le p$,

$$u_{k-p-l}^{(p)}-2u_{k-l}^{(p)}+u_{k+1-l}^{(p)}=0-2\cdot 2^{k-i-1}+2^{k-l}=0;$$

***(ii)*** $k-l>p$, let $k-l=p+t$, $t \ge 1$, then

$$u_{k-p-l}^{(p)}-2u_{k-l}^{(p)}+u_{k-l+1}^{(p)}=\sum_{j=1}^{t}F_j^{(p)}-2\sum_{j=1}^{p+t}F_j^{(p)}+\sum_{j=1}^{p+t+1}F_j^{(p)}$$

$$=-\sum_{j=t+1}^{p+t}F_j^{(p)}+F_{p+t+1}^{(p)}=0. \ \square$$

***Proof of the Main Result:*** Since $A_p$ is a square matrix, it is sufficient to prove that $A_p'$ is a right sided inverse. Using Lemma 2, we will prove statement (2). If we set $A_p=[c_{ij}]$, then

$$c_{ij}=\begin{cases}2, & j=i,\\ -1, & j=i+1 \text{ or } j=i-p,\\ 0, & \text{otherwise.}\end{cases}$$

For $k=1$ and $j \in \{1,2,\ldots,n-1\}$, we have

$$(A_pA_p')_{1j}=\sum_{l=1}^{n-1}c_{1l}a_{lj}=2a_{1j}-a_{2j}$$

$$=\frac{u_{n-j}^{(p)}}{u_n^{(p)}}\left(2u_1^{(p)}-u_2^{(p)}\right)+\left(-2u_{1-j}^{(p)}+u_{2-j}^{(p)}\right).$$

Using (1) and the definition of $F_j^{(p)}$,

$$2u_1^{(p)}-u_2^{(p)}=2F_1^{(p)}-(F_1^{(p)}+F_2^{(p)})=0$$

and

$$-2u_{1-j}^{(p)}+u_{2-j}^{(p)}=u_{2-j}^{(p)}=\begin{cases}u_1^{(p)}=1, & j=1,\\ 0, & j \in \{2,\ldots,n-1\}.\end{cases}$$

Therefore, $(A_pA_p')_{1j}=\delta_{1j}$ for $j \in \{1,2,\ldots,n-1\}$.

For $k \in \{2,\ldots,p\}$ and $j \in \{1,2,\ldots,n-1\}$, we have

$$(A_pA_p')_{kj}=\sum_{l=1}^{n-1}c_{kl}a_{lj}=2a_{kj}-a_{k+1,j}$$

$$=\frac{u_{n-j}^{(p)}}{u_n^{(p)}}\left(2u_k^{(p)}-u_{k+1}^{(p)}\right)+\left(-2u_{k-j}^{(p)}+u_{k+1-j}^{(p)}\right),$$

and from (1) and (3) it follows that

$$2u_k^{(p)} - u_{k+1}^{(p)} = -\left(u_{k-p}^{(p)} - 2u_k^{(p)} + u_{k+1}^{(p)}\right) = -\delta_{k0} = 0$$

and

$$-2u_{k-j}^{(p)} + u_{k+1-j}^{(p)} = u_{k-p-j}^{(p)} - 2u_{k-j}^{(p)} + u_{k+1-j}^{(p)} = \delta_{kj}.$$

For $k \in \{p+1, \ldots, n-2\}$ and $j \in \{1, 2, \ldots, n-1\}$, let $k = p+t$, $t \geq 1$. Then

$$\begin{aligned}
(A_p A_p')_{kj} &= \sum_{l=1}^{n-1} c_{kl} a_{lj} = c_{p+t,\,t} a_{tj} + c_{kk} a_{kj} + c_{k,k+1} a_{k+1,\,j} \\
&= -a_{tj} + 2a_{kj} - a_{k+1,\,j} = -a_{k-p,\,j} + 2a_{kj} - a_{k+1,\,j} \\
&= \frac{u_{n-j}^{(p)}}{u_n^{(p)}}\left[-u_{k-p}^{(p)} + 2u_k^{(p)} - u_{k+1}^{(p)}\right] + \left(u_{k-p-j}^{(p)} - 2u_{k-j}^{(p)} + u_{k+1-j}^{(p)}\right) \\
&= \frac{u_{n-j}^{(p)}}{u_n^{(p)}} \delta_{k0} + \delta_{kj} = \delta_{kj}.
\end{aligned}$$

For $k = n-1$ and $j \in \{1, 2, \ldots, n-1\}$, we have

$$\begin{aligned}
(A_p A_p')_{n-1,\,j} &= -a_{n-p-1,\,j} + 2a_{n-1,\,j} \\
&= -\frac{1}{u_n^{(p)}}\left(u_{n-p-1}^{(p)} u_{n-j}^{(p)} - u_n^{(p)} u_{n-p-1-j}^{(p)}\right) + \frac{2}{u_n^{(p)}}\left(u_{n-1}^{(p)} u_{n-j}^{(p)} - u_n^{(p)} u_{n-1-j}^{(p)}\right) \\
&= -\frac{u_{n-j}^{(p)}}{u_n^{(p)}}\left(u_{n-p-1}^{(p)} - 2u_{n-1}^{(p)}\right) + u_{n-1-p-j}^{(p)} - 2u_{n-1-j}^{(p)} \\
&= -\frac{u_{n-j}^{(p)}}{u_n^{(p)}}\left(-u_n^{(p)} + \delta_{n1}\right) + \delta_{n-1,\,j} - u_{n-j}^{(p)} = \delta_{n-1,\,j}. \quad \square
\end{aligned}$$

Using the previous theorem, we can now easily find inverses for the following band matrices $A$, with $A^{-1} = [a_{ij}]$:

- For the matrix

$$A = \begin{bmatrix}
2 & \overbrace{0 & \cdot & \cdot & 0}^{p-1} & -1 & & & & & & & \\
-1 & 2 & 0 & \cdot & \cdot & 0 & -1 & & & & & & \\
 & \cdot & \cdot & \cdot & & & \cdot & \cdot & & & & & \\
 & & \cdot & \cdot & \cdot & & & \cdot & \cdot & & & & \\
 & & & -1 & 2 & 0 & \cdot & \cdot & 0 & -1 & & & \\
 & & & & -1 & 2 & 0 & \cdot & \cdot & 0 & -1 & & \\
 & & & & & -1 & 2 & 0 & \cdot & \cdot & 0 & & \\
 & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & & & & & \cdot & \cdot & \cdot & \cdot & & \\
 & & & & & & & & -1 & 2 & 0 & & \\
 & & & & & & & & & -1 & 2 & &
\end{bmatrix}_{(n-1)\times(n-1)},$$

$$a_{ij} = \frac{1}{u_n^{(p)}}\left(u_j^{(p)} u_{n-i}^{(p)} - u_n^{(p)} u_{j-i}^{(p)}\right), \quad i, j \in \{1, 2, \ldots, n-1\}.$$

- For the matrix

$$A=\begin{bmatrix} & & & & & -1 & \overbrace{0 \quad \cdot \quad \cdot \quad 0}^{p-1} & 2 \\ & & & & -1 & 0 & \cdot \quad \cdot \quad 0 & 2 & -1 \\ & & & \cdot & \cdot & & & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & & & \cdot & \cdot & \cdot \\ & -1 & 0 & \cdot & \cdot & 0 & 2 & -1 \\ -1 & 0 & \cdot & \cdot & 0 & 2 & -1 \\ 0 & \cdot & \cdot & 0 & 2 & -1 \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 2 & -1 \\ 2 & -1 \end{bmatrix}_{(n-1)\times(n-1)},$$

$$a_{ij}=\frac{1}{u_n^{(p)}}\left(u_i^{(p)}u_j^{(p)}-u_n^{(p)}u_{i+j-n}^{(p)}\right),\quad i,j\in\{1,2,\dots,n-1\}.$$

- For the matrix

$$A=\begin{bmatrix} & & & & & & & & & -1 & 2 \\ & & & & & & & & -1 & 2 & 0 \\ & & & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot & & \cdot \\ & & & & & -1 & 2 & 0 & \cdot & \cdot & 0 \\ & & & & -1 & 2 & 0 & \cdot & \cdot & 0 & -1 \\ & & & -1 & 2 & 0 & \cdot & \cdot & 0 & -1 \\ & & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ -1 & 2 & 0 & & & 0 & -1 \\ 2 & \underbrace{0 \quad \cdot \quad \cdot \quad 0}_{p-1} & -1 \end{bmatrix}_{(n-1)\times(n-1)},$$

$$a_{ij}=\frac{1}{u_n^{(p)}}\left(u_{n-i}^{(p)}u_{n-j}^{(p)}-u_n^{(p)}u_{n-i-j}^{(p)}\right),\quad i,j\in\{1,2,\dots,n-1\}.$$

## REFERENCES


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