ON PALINDROMIC SEQUENCES FROM IRRATIONAL NUMBERS

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INTRODUCTION

A *palindrome* is a finite sequence $(x_1, x_2, \dots, x_n)$ of numbers satisfying

$$(x_1, x_2, \dots, x_n) = (x_n, x_{n-1}, \dots, x_1).$$

Let $\Delta_n = \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor$ for some positive irrational α, and $n = 1, 2, \dots$. In [2], Kimberling shows that there are infinitely many palindromes $(\Delta_1, \dots, \Delta_l)$ in the infinite Δ-sequence (or the *characteristic* word of the *Beatty sequence*). For example, for $\alpha = (1+\sqrt{5})/2$, the Δ-sequence begins 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, ... So $(\Delta_1, \dots, \Delta_l)$ is a palindrome for

$$l \in \{1, 3, 8, 21, 55, 144, 377, 987, \dots\},$$

and $(\Delta_2, \dots, \Delta_{l-1})$ is a palindrome for

$$l \in \{3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \dots\}.$$

(The examples in [2] only partly match this observation.) In [1] Droubay proves that, if $\alpha = (1+\sqrt{5})/2$, the number of palindromes of length n is exactly 1 if n is even, and 2 if n is odd (see also [3], e.g.). Then, how can we describe all the palindromes in the Δ-sequence? This paper gives an answer to this question.

MAIN RESULTS

As usual, we denote the continued fraction expansion of α by $\alpha = [a_0; a_1, a_2, \dots]$. Then its n^{th} (*total*) convergent $p_n / q_n = [a_0; a_1, \dots, a_n]$ is given by the recurrence relations

$$p_n = a_n p_{n-1} + p_{n-2} \quad (n = 0, 1, \dots), \quad p_{-2} = 0, \quad p_{-1} = 1,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n = 0, 1, \dots), \quad q_{-2} = 1, \quad q_{-1} = 0.$$

Define the n^{th} *intermediate* (or *partial*) *convergents* by $p_{n,r}/q_{n,r}$ $(r = 0, 1, 2, \dots, a_n - 1)$, where $p_{n,r} = rp_{n+1} + p_n$ and $q_{n,r} = rq_{n+1} + q_n$ ([3], cf. [5]). So, $p_{n,a_{n+2}} = p_{n+2}$ and $q_{n,a_{n+2}} = q_{n+2}$.

We define the fractional part of x by $\{x\} = x - \lfloor x \rfloor$.

Lemma 1: Let l and m be integers satisfying $l \ge 2m-1$. Then $(\Delta_m, \Delta_{m+1}, \dots, \Delta_{l-m+1})$ is a palindrome if and only if $\{k\alpha\} + \{(l-k)\alpha\}$ is invariant of k for $k = m-1, m, \dots, \lfloor (l+1)/2 \rfloor$.

Proof: By definition, $(\Delta_m, \Delta_{m+1}, \dots, \Delta_{l-m+1})$ is a palindrome if and only if, for $k = m-1$, m, ..., $\lfloor (l+1)/2 \rfloor$,

$$\lfloor (k+1)\alpha \rfloor + \lfloor (l-k-1)\alpha \rfloor = \lfloor k\alpha \rfloor + \lfloor (l-k)\alpha \rfloor,$$

or

$$\{(k+1)\alpha\} + \{(l-k-1)\alpha\} = \{k\alpha\} + \{(l-k)\alpha\}.$$

Of course, this also holds for $k = \lfloor (l+1)/2 \rfloor + 1$, $\lfloor (l+1)/2 \rfloor + 2, \dots, l-m$.

Lemma 2 (cf. Theorem 1, [2]): Let q be an integer with $q > q_1$. There are integers n and r with $n = 0, 1, \ldots$ and $r = 1, 2, \ldots, a_{n+2}$ such that $q = q_{n,r}$ if and only if, for $k = 1, 2, \ldots, q-1$, the sum $\{k\alpha\} + \{(q-k)\alpha\}$ is invariant of k, that is,

$$\{k\alpha\} + \{(q-k)\alpha\} = \begin{cases} \{q\alpha\}+1 & \text{if } n \text{ is even,} \\ \{q\alpha\} & \text{if } n \text{ is odd.} \end{cases}$$

Sublemma (Theorem 3.3, [5]): Let $q = 1, 2, \ldots, N-1$. If $q_{n,r-1} < N \le q_{n,r}$ $(2 \le r \le a_{n+2},\ n \ge 0)$, then

$$\{q_{n,r-1}\alpha\} \le \{q\alpha\} \le \{q_{n+1}\alpha\} \quad \text{if } n \text{ is even,}$$
$$\{q_{n+1}\alpha\} \le \{q\alpha\} \le \{q_{n,r-1}\alpha\} \quad \text{if } n \text{ is odd.}$$

If $q_{n+1} < N \le q_{n,1}$ $(n \ge 0)$, then

$$\{q_n\alpha\} \le \{q\alpha\} \le \{q_{n+1}\alpha\} \quad \text{if } n \text{ is even,}$$
$$\{q_{n+1}\alpha\} \le \{q\alpha\} \le \{q_n\alpha\} \quad \text{if } n \text{ is odd.}$$

If $N \le q_1$, then $\{\alpha\} < \{2\alpha\} < \cdots < \{(N-1)\alpha\}$.

Proof of Lemma 2: If $q = q_{n,r}$ for some integers n and r, then by the Sublemma for $k = 1, 2, \ldots, q-1$,

$$\{k\alpha\} > \{q\alpha\} \quad \text{if } n \text{ is even,}$$
$$\{k\alpha\} < \{q\alpha\} \quad \text{if } n \text{ is odd.}$$

Thus, for $k = 1, 2, \ldots, q-1$,

$$\{k\alpha\} + \{(q-k)\alpha\} > \{q\alpha\} \quad \text{if } n \text{ is even,}$$
$$\{k\alpha\} + \{(q-k)\alpha\} < \{q\alpha\}+1 \quad \text{if } n \text{ is odd.}$$

Therefore, for $k = 1, 2, \ldots, q-1$,

$$\{k\alpha\} + \{(q-k)\alpha\} = \begin{cases} \{q\alpha\}+1 & \text{if } n \text{ is even,} \\ \{q\alpha\} & \text{if } n \text{ is odd.} \end{cases}$$

Because $\{k\alpha\} + \{(q-k)\alpha\}$ takes only the values $\{q\alpha\}$ or $\{q\alpha\}+1$, the sum $\{k\alpha\} + \{(q-k)\alpha\}$ is invariant of k.

On the other hand, if $q \ne q_{n,r}$ for some integers n and r, then there exist integers k' and k'' with $k' \ne k''$ and $0 < k',\ k'' < q$ such that $\{k'\alpha\} < \{q\alpha\} < \{k''\alpha\}$. Hence,

$$\{k'\alpha\} + \{(q-k')\alpha\} < \{q\alpha\}+1 \quad \text{and} \quad \{k''\alpha\} + \{(q-k'')\alpha\} > \{q\alpha\}.$$

Since $\{k\alpha\} + \{(q-k)\alpha\}$ takes only the values $\{q\alpha\}$ or $\{q\alpha\}+1$, the sum is not invariant of k for $k = 1, 2, \ldots, q-1,$.

When $m = 2$, we have the first main theorem by using Lemmas 1 and 2.

Theorem 1: Let the continued fraction expansion of an irrational α be

$$\alpha = [a_0; a_1, a_2, \ldots, a_n, \ldots].$$

Then $(\Delta_2, \ldots, \Delta_{l-1})$ is a palindrome only for

$$l \in \{\underbrace{1, 2, \ldots, q_1}_{a_1}, \underbrace{q_1+1, 2q_1+1, \ldots, q_2}_{a_2}, \underbrace{q_2+q_1, 2q_2+q_1, \ldots, q_3}_{a_3}, \ldots, \underbrace{q_{n-1}+q_{n-2}, 2q_{n-1}+q_{n-2}, \ldots, q_n}_{a_n}, \ldots\} - \{1, 2\}.$$

Proof: Since $1/(a_1+1) < \{\alpha\} = [0; a_1, a_2, \ldots] < 1/a_1$, we have, for $a_1 \ge 2$,

$$\Delta_2 + \cdots + \Delta_{q_1} = \lfloor a_1\alpha \rfloor - \lfloor \alpha \rfloor = a_1\lfloor \alpha \rfloor - \lfloor \alpha \rfloor = (a_1 - 1)\lfloor \alpha \rfloor,$$

yielding $\Delta_2 = \cdots = \Delta_{q_1} = \lfloor \alpha \rfloor$ because $\Delta_n = \lfloor \alpha \rfloor$ or $\lfloor \alpha \rfloor + 1$. Hence, $(\Delta_2, \ldots, \Delta_{l-1})$ is a palindrome for $l = 3, 4, \ldots, q_1 + 1$. For $a_1 = 1$, it is trivial that $l = 3$.

Set $n = 0, 1, 2, \ldots$. By Lemma 2 for $k = 1, 2, \ldots, q_{n,r} - 2$ $(r = 1, 2, \ldots, a_{n+2})$,

$$\{(k+1)\alpha\} + \{(q_{n,r} - (k+1))\alpha\} = \{k\alpha\} + \{(q_{n,r} - k)\alpha\}.$$

Thus, by Lemma 1, $(\Delta_2, \ldots, \Delta_{l-1})$ is a palindrome for $l = q_{n,r}$ $(r = 1, 2, \ldots, a_{n+2})$. Lemma 2 also shows that there is no other possibility for l.

Example 1: Let $\alpha = e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]$. Then the denominators of its convergents are

$$(q_1, q_2, q_3, \ldots, q_{10}, \ldots) = (1, 3, 4, 7, 32, 39, 71, 465, 536, 1001, \ldots).$$

Hence, $(\Delta_2, \ldots, \Delta_{l-1})$ is a palindrome for

$$\begin{aligned} l \in \{&\underbrace{1}_{1}, \underbrace{2, 3}_{2}, \underbrace{4}_{1}, \underbrace{7}_{1}, \underbrace{11, 18, 25, 32}_{4}, \underbrace{39}_{1}, \\ &\underbrace{71}_{1}, \underbrace{110, 181, 252, 323, 394, 465}_{6}, \underbrace{536}_{1}, \underbrace{1001}_{1}, \ldots\} - \{1, 2\} \\ = \{&3, 4, 7, 11, 18, 25, 32, 39, 71, 110, 181, 252, 323, 394, 465, 536, 1001, \ldots\}. \end{aligned}$$

In fact, Δ begins with $2, \check{3}, \hat{3}, 2, 3, \hat{3}, 3, 2, 3, \hat{3}, 2, 3, 3, 3, 2, 3, \hat{3}, 2, 3, 3, 3, 2, 3, \hat{3}, 2, 3, 3, 3, 2, 3, \hat{3}, 2, \ldots$. One can see the palindromes between ˇ and ˆ (included).

Next, we put $m = 1$ to obtain the following result.

Theorem 2: $(\Delta_1, \ldots, \Delta_l)$ is a palindrome only for

$$l \in \{\underbrace{1, 2, \ldots, q_1}_{a_1}, \underbrace{q_2+q_1, 2q_2+q_1, \ldots, q_3}_{a_3}, \underbrace{q_4+q_3, 2q_4+q_3, \ldots, q_5}_{a_5}, \ldots, \underbrace{q_{2n}+q_{2n-1}, 2q_{2n}+q_{2n-1}, \ldots, q_{2n+1}}_{a_{2n+1}}, \dot{s}\}.$$

Proof: Since $\Delta_1 = \Delta_2 = \cdots = \Delta_{q_1} = \lfloor \alpha \rfloor$, $(\Delta_1, \ldots, \Delta_l)$ is a palindrome for $l = 1, 2, \ldots, q_1$. Set $n = 0, 1, 2, \ldots$. By Lemma 2 for $k = 2, 3, \ldots, q_{n,r} - 1$ $(r = 1, 2, \ldots, a_{n+2})$,

$$\{k\alpha\} + \{(q_{n,r} - k)\alpha\} = \{(k-1)\alpha\} + \{(q_{n,r} - k + 1)\alpha\}.$$

And for $k = 1$, $\{\alpha\} + \{(q_{n,r} - 1)\alpha\} = \{q_{n,r}\alpha\}$ is true only when n is odd. Therefore, $(\Delta_2, \ldots, \Delta_{l-1})$ is a palindrome for $l = q_{2n-1,r}$ $(r = 1, 2, \ldots, a_{2n+1};\ n = 1, 2, \ldots)$. By Lemma 2, all the possibilities for l appear here.

MORE PALINDROMES

There are infinitely many palindromes that do not start from Δ_1 or Δ_2 in the Δ-sequence. In other words, for any integer m, there exist infinitely many integers l with $l \geq 2m-1$ such that

$$(\Delta_m, \Delta_{m+1}, \ldots, \Delta_{l-m+1})$$

is palindromic. Defining $\Delta_0 = \lfloor 0\alpha \rfloor - \lfloor -\alpha \rfloor$, we have the following theorem.

Theorem 3: $(\Delta_0, \Delta_1, \ldots, \Delta_{l+1})$ is a palindrome only for

$$l \in \{q_1, \underbrace{q_2+q_1, 2q_2+q_1, \ldots, q_3}_{a_3}, \underbrace{q_4+q_3, 2q_4+q_3, \ldots, q_5}_{a_5}, \ldots, \underbrace{q_{2n}+q_{2n-1}, 2q_{2n}+q_{2n-1}, \ldots, q_{2n+1}}_{a_{2n+1}}, \ldots\}.$$

Proof: Since $\Delta_0 = -\lfloor -\alpha \rfloor = \lfloor \alpha \rfloor + 1 = \Delta_{q_1+1}$ and $\Delta_1 = \Delta_2 = \cdots = \Delta_{q_1} = \lfloor \alpha \rfloor$, $(\Delta_0, \ldots, \Delta_{l+1})$ is a palindrome for $l = q_1$. By Lemma 2,

$$\{(k-1)\alpha\} + \{(q_{n,r}-k+1)\alpha\} = \{(k-2)\alpha\} + \{(q_{n,r}-k+2)\alpha\}$$

holds for $k = 3, 4, \ldots, q_{n,r}-1$. For $k=2$, $\{\alpha\} + \{(q_{n,r}-1)\alpha\} = \{q_{n,r}\alpha\}$ is true only when n is odd.

Consider the case $k=1$. When n is odd,

$$\begin{aligned}\{q_{n,r}\alpha\} + \{\alpha\} &= q_{n,r}\alpha - \lfloor q_{n,r}\alpha \rfloor + \{\alpha\} \\ &= 1 + \{\alpha\} - (p_{n,r} - q_{n,r}\alpha) > 1 + \frac{1}{a_1+1} - \frac{1}{q_{n+1}} \geq 1.\end{aligned}$$

Therefore, $\{q_{n,r}\alpha\} + \{\alpha\} = \{(q_{n,r}+1)\alpha\} + 1$ or $\{q_{n,r}\alpha\} = \{-\alpha\} + \{(q_{n,r}+1)\alpha\}$. Of course, there are no other possibilities for l.

Next, we shall consider the cases where $m \geq 3$. From Theorem 1, we immediately obtain the following.

Corollary: For $m = 3, 4, \ldots, (\Delta_m, \Delta_{m+1}, \ldots, \Delta_{l-m+1})$ is a palindrome for

$$l \in \{\underbrace{1, 2, \ldots, q_1}_{a_1}, \underbrace{q_1+1, 2q_1+1, \ldots, q_2}_{a_2}, \underbrace{q_2+q_1, 2q_2+q_1, \ldots, q_3}_{a_3}, \ldots, \underbrace{q_{n-1}+q_{n-2}, 2q_{n-1}+q_{n-2}, \ldots, q_n}_{a_n}, \ldots\}$$

with $l \geq 2m-1$.

However, this does not necessarily show all the palindromes. If $\{k\alpha\} + \{(l-k)\alpha\}$ is invariant of k just for $k = m-1, m, \ldots, \lfloor (l+1)/2 \rfloor$, $(\Delta_m, \Delta_{m+1}, \ldots, \Delta_{l-m+1})$ already becomes a palindrome. For example, when $m=3$, all the palindromes are described as follows.

Theorem 4: $(\Delta_3, \Delta_4, \ldots, \Delta_{l-2})$ is a palindrome only for

$$l \in \{\underbrace{1, 2, \ldots, q_1}_{a_1}, \underbrace{q_1+1, 2q_1+1, \ldots, q_2}_{a_2}, \underbrace{q_2+q_1, 2q_2+q_1, \ldots, q_3}_{a_3}, \ldots, \underbrace{q_{n-1}+q_{n-2}, 2q_{n-1}+q_{n-2}, \ldots, q_n}_{a_n}, \ldots\}$$

with $l \geq 5$, or

$$l = q_1 + 2 \text{ if } a_1 \geq 3; \quad l = q_2 + 2 \text{ if } a_1 = 1 \text{ and } a_2 \leq 2.$$

Proof: Let n be even. By Lemma 2, if $\{\alpha\} + \{(q-1)\alpha\} = \{q\alpha\}$ and, for $k = 2, 3, \ldots, q-2$, $\{k\alpha\} + \{(q-k)\alpha\} = \{q\alpha\} + 1$, then $(\Delta_3, \Delta_4, \ldots, \Delta_{q-2})$ is a palindrome. Therefore, $\{\alpha\} < \{q\alpha\}$ or $\{(q-1)\alpha\} < \{q\alpha\}$, and $\{k\alpha\} > \{q\alpha\}$ $(k = 2, 3, \ldots, q-2)$.

If $q < q_1$, this is clearly impossible.

If $q_{n+1} < q < q_{n,1}$, then, by the Sublemma, $\{q_n\alpha\} < \{q\alpha\} < \{q_{n+1}\alpha\}$. So, $q_n = 1$ or $q_n = q-1$. But $q_n = 1$ is impossible because $q \geq 5$. The case $q = q_n + 1$ does not satisfy $q > q_{n+1}$.

If $q_{n,r-1} < q < q_{n,r}$ for some integers n and $r \geq 2$, then, by the Sublemma, $\{q_{n,r-1}\alpha\} < \{q\alpha\} < \{q_{n,r}\alpha\}$. So, $q_{n,r-1} = 1$ or $q_{n,r-1} = q-1$. But $q_{n,r-1} = 1$ is impossible because $q \geq 5$. Suppose that $q_{n,r-1} = q-1$. Since

$$\{q_{n,r-1}\alpha\} < \{((r-2)q_{n+1} + q_n)\alpha\} < \{(q_{n,r-1}+1)\alpha\} = \{q\alpha\},$$

we must have $(r-2)q_{n+1} + q_n = 1$, yielding $r = 2$. Hence, $n = 0$. Similarly, we have $n = 1$ and $a_1 = 1$ when n is odd. Therefore, $q = q_{0,1} + 1 = q_1 + 2$ if $a_1 \geq 3$; $q = q_{1,1} + 1 = q_2 + 2$ if $a_1 = 1$ and $a_2 \geq 2$.

But it is not so easy to describe all the palindromes for general $m \geq 3$. It is convenient to use the following Lemma to find the extra palindromes in addition to those appearing in the Corollary.

Lemma 3: Let $q \neq q_{n,r}$ for any integers n and r. Suppose that the sequence $\{\alpha\}$, $\{2\alpha\}$, ..., $\{q\alpha\}$ is sorted as

$$\{u_1\alpha\} < \{u_2\alpha\} < \cdots < \{u_k\alpha\} < \{q\alpha\} < \{u_{k+1}\alpha\} < \cdots < \{u_{q-1}\alpha\},$$

where $\{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_{q-1}\} = \{1, 2, \ldots, q-1\}$. Put

$$M = \max_{i \leq j \leq k} \min(u_j, q-u_j) \quad \text{and} \quad M' = \max_{k+1 \leq j \leq q-1} \min(u_j, q-u_j).$$

If $q \geq 2M+3$, then $(\Delta_m, \ldots, \Delta_{q-m+1})$ is palindromic with $m = M+2, M+3, \ldots, \lfloor (q+1)/2 \rfloor$.

If $q \geq 2M'+3$, then $(\Delta_m, \ldots, \Delta_{q-m+1})$ is palindromic with $m = M'+2, M'+3, \ldots, \lfloor (q+1)/2 \rfloor$.

Remark: The conditions $q \geq 2M+3$ and $q \geq 2M'+3$ do not hold simultaneously. For, either $M = q/2$ or $M' = q/2$ when q is even; either $M = (q-1)/2$ or $M' = (q-1)/2$ when q is odd. It is possible that both conditions fail for some q's.

Proof: First of all, notice that $\{k\alpha\}$ and $\{(q-k)\alpha\}$ lie on the same side of $\{q\alpha\}$. If $\{k\alpha\} < \{q\alpha\} < \{(q-k)\alpha\}$, then $\{q\alpha\} < \{k\alpha\} + \{(q-k)\alpha\} < \{q\alpha\} + 1$, yielding a contradiction because $\{k\alpha\} + \{(q-k)\alpha\}$ must be either $\{q\alpha\}$ or $\{q\alpha\} + 1$. Now, since $\{M\alpha\} < \{q\alpha\} < \{k\alpha\}$ $(k = M+1, M+2, \ldots, \lfloor (q+1)/2 \rfloor$, we have

$$\{M\alpha\} + \{(q-M)\alpha\} < \{q\alpha\} + 1 \quad \text{and}$$
$$\{k\alpha\} + \{(q-k)\alpha\} > \{q\alpha\} \; (k = M+1, M+2, \ldots, \lfloor (q+1)/2 \rfloor),$$

yielding

$$\{M\alpha\} + \{(q-M)\alpha\} = \{q\alpha\} \quad \text{and}$$
$$\{k\alpha\} + \{(q-k)\alpha\} = \{q\alpha\} + 1 \; (k = M+1, M+2, \ldots, \lfloor (q+1)/2 \rfloor).$$

Together with Lemma 1 we have the desired result. The proof for M' is similar and is omitted here.

Example 2: Let $\alpha = (\sqrt{29}+5)/2 = [5; 5, 5, 5, \ldots]$. Then the sequence $\{\alpha\}$, $\{2\alpha\}$, ..., $\{483\alpha\}$ is sorted as

$$\begin{aligned}\{431\alpha\} &< \{296\alpha\} < \{161\alpha\} < \{26\alpha\} < \{457\alpha\} < \{322\alpha\}\\ &< \{187\alpha\} < \{52\alpha\} < \{483\alpha\} < \underbrace{\cdots\cdots\cdots}_{\text{all the others}} < \{462\alpha\}\\ &< \{327\alpha\} < \{192\alpha\} < \{57\alpha\} < \{353\alpha\} < \{218\alpha\} < \{83\alpha\}\\ &< \{379\alpha\} < \{244\alpha\} < \{109\alpha\} < \{405\alpha\} < \{270\alpha\} < 135\alpha\}.\end{aligned}$$

When $q = 483$, $M = \max(52, 187, 161, 26) = 187$, and $q \geq 2M+3$. By Lemma 3, $(\Delta_m, \Delta_{m+1}, \Delta_{q-m+1})$ is palindromic for $q = 483$ with $m = 189, 190, \ldots, 242$ only. Of course, $M' = 241$ does not satisfy the condition $q \geq 2M'+3$.

When $q = 462$, $M = \max(135, 192, 57, 109, 218, 83) = 218$, and $q \geq 2M+3$. By Lemma 3, $(\Delta_m, \Delta_{m+1}, \Delta_{q-m+1})$ is palindromic only for $q = 462$ with $m = 220, 221, \ldots, 231$.

HOW TO FIND M OR M' IN LEMMA 3

Lemma 3 shows that once M or M' is given for an arbitrary positive integer q with $q \neq q_{n,r}$, all the palindromes $(\Delta_m, \ldots, \Delta_{q-m+1})$ can be discovered without omission. It is, however, tiresome to sort the sequence $\{\alpha\}, \{2\alpha\}, \ldots, \{q\alpha\}$ as seen in Example 2. In fact, M or M' can be determined without any real sorting.

Consider the general integer q with $q \neq q_{n,i}$ for arbitrary integers n and i. For example, put $q = rq_{n+1} + jq_n$ $(r = 1, 2, \ldots, a_{n+2};\ j = 2, 3, \ldots, a_{n+1})$. Then, since

$$\begin{aligned}&\{(rq_{n+1}+q_n)\alpha\} < \cdots < \{(q_{n+1}+q_n)\alpha\} < \{q_n\alpha\}\\ &< \{(rq_{n+1}+2q_n)\alpha\} < \cdots < \{(q_{n+1}+2q_n)\alpha\} < \{2q_n\alpha\} < \cdots\\ &< \{(rq_{n+1}+jq_n)\alpha\} < \cdots < \{(q_{n+1}+jq_n)\alpha\} < \{jq_n\alpha\} < \cdots\end{aligned}$$

when n is even (the order is reversed, and M' replaces M, when n is odd; cf. [5]). M in Lemma 3 can be determined by

$$M = \begin{cases}(r-1)q_{n+1}/2 + (j-1)q_n & \text{if } r \text{ is odd,}\\ (rq_{n+1}+(j-1)q_n)/2 & \text{if } r \text{ is even and } j \text{ is odd,}\\ (rq_{n+1}+jq_n)/2 & \text{if } r \text{ is even and } j \text{ is even.}\end{cases}$$

The condition in Lemma 3, $q \geq 2M+3$, is satisfied if $q_{n+1} \geq (j-2)q_n + 3$ (r: odd); $q_n \geq 3$ (r: even, j: odd). But this condition is never satisfied if r is even and j is even.

Similarly, for $q = rq_{n+1} + jq_n - iq_{n-1}$ $(r = 1, 2, \ldots, a_{n-2};\ j = 2, 3, \ldots, a_{n+1};\ i = 1, 2, \ldots, a_n)$, we have

$$M = \begin{cases}(rq_{n+1}+jq_n-(i+1)q_{n-1})/2 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: even;}\\ (rq_{n+1}+jq_n-iq_{n-1})/2 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: odd;}\\ (rq_{n+1}+(j-1)q_n-q_{n-1})/2 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 1 \pmod 2;\\ (rq_{n+1}+(j-1)q_n)/2 & \text{if } r\text{: even, } j\text{: odd;}\\ (rq_{n+1}+jq_n-iq_{n-1})/2 & \text{if } r\text{: even, } j\text{: even, } i\text{: even;}\\ (rq_{n+1}+jq_n-(i+1)q_{n-1})/2 & \text{if } r\text{: even, } j\text{: even, } i\text{: odd.}\end{cases}$$

And the condition $q \geq 2M+3$ is satisfied when

$$\begin{cases} q_{n-1} \geq 3 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: even;} \\ \text{never satisfied} & \text{if } r\text{: odd, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: odd;} \\ q_n \geq (i-1)q_{n-1}+3 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 1 \pmod 2; \\ q_n \geq iq_{n-1}+3 & \text{if } r\text{: even, } j\text{: odd;} \\ \text{never satisfied} & \text{if } r\text{: even, } j\text{: even, } i\text{: even;} \\ q_{n-1} \geq 3 & \text{if } r\text{: even, } j\text{: even, } i\text{: odd.} \end{cases}$$

Next, put $q = rq_{n+1} - jq_n$ $(r = 1, 2, \ldots, a_{n+2}+1;\ j = 0, 1, \ldots, a_{n+1})$. Then M in Lemma 3 can be determined by

$$M = \begin{cases} (r-1)q_{n+1}/2 & \text{if } r \text{ is odd,} \\ (rq_{n+1}-(j+1)q_n)/2 & \text{if } r \text{ is even and } j \text{ is odd,} \\ (rq_{n+1}-jq_n)/2 & \text{if } r \text{ is even and } j \text{ is even,} \end{cases}$$

because

$$\begin{aligned} &\{q_{n+1}\alpha\} < \{2q_{n+1}\alpha\} < \cdots < \{rq_{n+1}\alpha\} \\ &< \{(q_{n+1}-q_n)\alpha\} < \{(2q_{n+1}-q_n)\alpha\} < \cdots < \{(rq_{n+1}-q_n)\alpha\} \\ &< \{(q_{n+1}-2q_n)\alpha\} < \{(2q_{n+1}-2q_n)\alpha\} < \cdots < \{(rq_{n+1}-2q_n)\alpha\} < \cdots \\ &< \{(q_{n+1}-jq_n)\alpha\} < \{(2q_{n+1}-jq_n)\alpha\} < \cdots < \{(rq_{n+1}-jq_n)\alpha\} < \cdots \end{aligned}$$

when n is odd (the order is reversed, and M' replaces M, when n is even).

The condition $q \geq 2M+3$ is satisfied when

$$\begin{cases} q_{n+1} \geq jq_n + 3 & \text{if } r \text{ is odd,} \\ q_n \geq 3 & \text{if } r \text{ is even and } j \text{ is odd,} \\ \text{never satisfied} & \text{if } r \text{ is even and } j \text{ is even.} \end{cases}$$

Similarly, for $q = rq_{n+1} - jq_n + iq_{n-1}$ $(r = 1, 2, \ldots, a_{n+2}+1;\ j = 0, 1, \ldots, a_{n+1};\ i = 0, 1, \ldots, a_n)$, we have

$$M = \begin{cases} (rq_{n+1}-jq_n+(i-1)q_{n-1})/2 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: even;} \\ (rq_{n+1}-jq_n+iq_{n-1})/2 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: odd;} \\ (rq_{n+1}-(j+1)q_n+(2i-1)q_{n-1})/2 & \text{if } r\text{: odd, } j+a_{n+1} \equiv 1 \pmod 2; \\ (rq_{n+1}-(j+1)q_n+2iq_{n-1})/2 & \text{if } r\text{: even, } j\text{: odd;} \\ (rq_{n+1}-jq_n+iq_{n-1})/2 & \text{if } r\text{: even, } j\text{: even, } i\text{: even;} \\ (rq_{n+1}-jq_n+(i-1)q_{n-1})/2 & \text{if } r\text{: even, } j\text{: even, } i\text{: odd.} \end{cases}$$

The condition $q \geq 2M+3$ is satisfied when

$$\begin{cases} q_{n-1} \geq 3 & \text{if } r\text{: even, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: even;} \\ \text{never satisfied} & \text{if } r\text{: even, } j+a_{n+1} \equiv 0 \pmod 2,\ i\text{: odd;} \\ q_n \geq (i-1)q_{n-1}+3 & \text{if } r\text{: even, } j+a_{n+1} \equiv 1 \pmod 2; \\ q_n \geq iq_{n-1}+3 & \text{if } r\text{: odd, } j\text{: odd;} \\ \text{never satisfied} & \text{if } r\text{: odd, } j\text{: even, } i\text{: even;} \\ q_{n-1} \geq 3 & \text{if } r\text{: odd, } j\text{: even, } i\text{: odd.} \end{cases}$$

Generally speaking, M (or M') in Lemma 3 can be determined as follows.

Lemma 4: If $M = (\cdots - uq_{N+1} + (s-1)q_N)/2$ for $q = \cdots - uq_{N+1} + sq_N$, then $M = (\cdots - uq_{N+1} + (s-1)q_N)/2$ for $q = \cdots - uq_{N+1} + sq_N - tq_{N-1}$ $(t = 1, 2, \ldots, a_N)$.

If $M = (\cdots - uq_{N+1} + sq_N)/2$ for $q = \cdots - uq_{N+1} + sq_N$, then

$$M = \begin{cases} (\cdots - uq_{N+1} + sq_N - tq_{N-1})/2 & \text{if } t \text{ is even,} \\ (\cdots - uq_{N+1} + sq_N - (t+1)q_{N-1})/2 & \text{if } t \text{ is odd,} \end{cases}$$

for $q = \cdots - uq_{N+1} + sq_N - tq_{N-1}$.

If $M = (\cdots - (u+1)q_{N+1} + (2s-1)q_N)/2$ for $q = \cdots - uq_{N+1} + sq_N$, then

$$M = \begin{cases} (\cdots - uq_{N+1} + sq_N - tq_{N-1})/2 & \text{if } s + a_{N+1} \equiv 1 \pmod 2 \text{ and } t \text{ is odd,} \\ (\cdots - uq_{N+1} + sq_N - (t+1)q_{N-1})/2 & \text{if } s + a_{N+1} \equiv 1 \pmod 2 \text{ and } t \text{ is even,} \\ (\cdots - uq_{N+1} + (s-1)q_N - q_{N-1})/2 & \text{if } s + a_{N+1} \equiv 0 \pmod 2, \end{cases}$$

for $q = \cdots - uq_{N+1} + sq_N - tq_{N-1}$.

If $M = (\cdots - (u+1)q_{N+1} + 2sq_N)/2$ for $q = \cdots - uq_{N+1} + sq_N$, then

$$M = \begin{cases} (\cdots - uq_{N+1} + sq_N - tq_{N-1})/2 & \text{if } s + a_{N+1} \equiv 0 \pmod 2 \text{ and } t \text{ is odd,} \\ (\cdots - uq_{N+1} + sq_N - (t+1)q_{N-1})/2 & \text{if } s + a_{N+1} \equiv 0 \pmod 2 \text{ and } t \text{ is even,} \\ (\cdots - uq_{N+1} + (s-1)q_N - q_{N-1})/2 & \text{if } s + a_{N+1} \equiv 1 \pmod 2, \end{cases}$$

for $q = \cdots - uq_{N+1} + sq_N - tq_{N-1}$.

Lemma 4': If $M = (\cdots + uq_{N+1} - (s+1)q_N)/2$ for $q = \cdots + uq_{N+1} - sq_N$, then $M = (\cdots + uq_{N+1} - (s+1)q_N + 2tq_{N-1})/2$ for $q = \cdots + uq_{N+1} - sq_N + tq_{N-1}$ $(t = 1, 2, \ldots, a_N)$.

If $M = (\cdots + uq_{N+1} - sq_N)/2$ for $q = \cdots + uq_{N+1} - sq_N$, then

$$M = \begin{cases} (\cdots + uq_{N+1} - sq_N + tq_{N-1})/2 & \text{if } t \text{ is even,} \\ (\cdots + uq_{N+1} - sq_N + (t-1)q_{N-1})/2 & \text{if } t \text{ is odd,} \end{cases}$$

for $q = \cdots + uq_{N+1} - sq_N + tq_{N-1}$.

If $M = (\cdots + (u-1)q_{N+1} - q_N)/2$ for $q = \cdots + uq_{N+1} - sq_N$, then

$$M = \begin{cases} (\cdots + uq_{N+1} - sq_N + tq_{N-1})/2 & \text{if } s + a_{N+1} \equiv 1 \pmod 2 \text{ and } t \text{ is odd,} \\ (\cdots + uq_{N+1} - sq_N + (t-1)q_{N-1})/2 & \text{if } s + a_{N+1} \equiv 1 \pmod 2 \text{ and } t \text{ is even,} \\ (\cdots + uq_{N+1} - (s+1)q_N + (2t-1)q_{N-1})/2 & \text{if } s + a_{N+1} \equiv 0 \pmod 2, \end{cases}$$

for $q = \cdots + uq_{N+1} - sq_N + tq_{N-1}$.

If $M = (\cdots + (u-1)q_{N+1})/2$ for $q = \cdots + uq_{N+1} - sq_N$, then

$$M = \begin{cases} (\cdots + uq_{N+1} - sq_N + tq_{N-1})/2 & \text{if } s + a_{N+1} \equiv 0 \pmod 2 \text{ and } t \text{ is odd,} \\ (\cdots + uq_{N+1} - sq_N + (t-1)q_{N-1})/2 & \text{if } s + a_{N+1} \equiv 0 \pmod 2 \text{ and } t \text{ is even,} \\ (\cdots + uq_{N+1} - (s+1)q_N + (2t-1)q_{N-1})/2 & \text{if } s + a_{N+1} \equiv 1 \pmod 2, \end{cases}$$

for$q = \cdots + uq_{N+1} - sq_N + tq_{N-1}$.

Example 3: There is a reason for our providing two alternative expressions for each integer q. For instance, let

$$\alpha = \frac{\sqrt{29}+5}{2} = [5; 5, 5, 5, \ldots].$$

For $q = 3q_2 - q_1 + 4q_0 = 3 \cdot 26 - 5 + 4 \cdot 1 = 77$, we have $M = (3q_2 - q_1 + 3q_0)/2 = 38$, not satisfying $q \geq 2M + 3$. However, for $q = 2q_2 + 5q_1 = 77$, we obtain $M' = (2q_2 + 4q_1)/2 = 36$, satisfying $q \geq 2M' + 3$ and leading to the conclusion that $(\Delta_m, \ldots, \Delta_{q-m+1})$ is palindromic for $q = 77$ with $m = 38$ and 39.

SUMMARY

When $q = q_{n,r}$, the palindromic sequences $(\Delta_m, \ldots, \Delta_{q-m+1})$ can be found by Theorem 1, 2, 3, 4, or the Corollary. When $q \neq q_{n,r}$, all the other palindromes can be discovered by Lemma 3 with Lemma 4 and Lemma 4′.

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