ON THE FREQUENCY OF OCCURRENCE OF α^i IN THE α-EXPANSIONS OF THE POSITIVE INTEGERS

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1. INTRODUCTION

Most students are familiar with representations of integers using various integral bases. In [1], George Bergman introduced a system using the irrational base $\alpha = \frac{1+\sqrt{5}}{2}$. The number α is of course the well-known golden ratio,* often defined as the limit of the sequence $\{F_n / F_{n-1}\}$, where F_n is the n^{th} Fibonacci number. Under this system, we can represent any natural number n (uniquely) as the sum of nonconsecutive powers of α. This means that, for any natural number n, there exists a unique sequence $\{e_i\}$, where $e_i \in \{0, 1\}$ for all i, such that $n = \sum_{i=-\infty}^{\infty} e_i \alpha^i$ and $e_i e_{i+1} = 0$ for each i. The *α-expansion of n* is $\ldots e_{-2}e_{-1}\underline{e_0}e_1e_2\ldots$, where we adopt the convention of underlining the zero$^{\text{th}}$ coordinate and omitting leading and trailing zeros when convenient. For example, $5 = \alpha^{-4} + \alpha^{-1} + \alpha^3$, so the base-$\alpha$ representation of 5 is $1001\underline{0}001$. Table 1 shows the α-expansions of the first 30 natural numbers. Table 2 shows the base 2 representations.

If we look down any column of the base 2 representations, it is easy to detect the patterns, which involve strings of 0s and 1s of equal length, so that the ratio of 1s to 0s is almost 1. The situation for other positive integral bases is analogous. In contrast, the columns in the α-expansions also exhibit patterns, but these are not so easy to detect or describe. The purpose of this paper is to explore some of these patterns. For each positive integer n, let $Ratio_i(n)$ be the ratio of the numbers $k \leq n$ that do have α^i in their α-expansions to those that do not. In other words, $Ratio_i(n)$ is the ratio of 1s to 0s in the i^{th} column (i.e., the column corresponding to α^i) of the α-expansions of the integers 1 through n.

Hart and Sanchis showed in [6] that $Ratio_0(n) \to \alpha^{-2}$ as $n \to \infty$, thus proving Conjecture 1 from [2], as well as answering a question posed by Bergman in [1]. In this paper, we generalize the techniques used in [6] to derive the behavior of $Ratio_i(n)$ for all other values of i. It should come as no surprise that α-expansions are closely related to the Fibonacci sequence. Indeed, any natural number n can be expressed uniquely as the sum of Fibonacci numbers F_k (here $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$). This is the well-known Zeckendorf decomposition of n. Grabner et al. ([3], [4]) showed that, for $m \geq \log_m k$, the Zeckendorf decomposition of kF_m can be produced by replacing each α^i in the α-expansion of k with F_{m+i}. Thus, our results also provide information about the occurrence of F_{k+i} in the Zeckendorf decomposition of kF_k.

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* In [5] and [6], the symbol β was used for this quantity; we have decided to change to the more commonly used α.

TABLE 1. α-Expansions of the Integers 1-30

n																
1									1							
2							1	0	0	1						
3							1	0	0	0	1					
4							1	0	1	0	1					
5					1	0	0	1	0	0	0	1				
6					1	0	0	0	0	1	0	1				
7					1	0	0	0	0	0	0	0	1			
8					1	0	0	0	1	0	0	0	1			
9					1	0	1	0	0	1	0	0	1			
10					1	0	1	0	0	0	1	0	1			
11					1	0	1	0	1	0	1	0	1			
12			1	0	0	1	0	1	0	0	0	0	0	1		
13			1	0	0	1	0	0	0	1	0	0	0	1		
14			1	0	0	1	0	0	0	0	1	0	0	1		
15			1	0	0	1	0	0	1	0	1	0	0	1		
16			1	0	0	0	0	1	0	0	0	1	0	1		
17			1	0	0	0	0	0	0	1	0	1	0	1		
18			1	0	0	0	0	0	0	0	0	0	0	0	1	
19			1	0	0	0	0	0	1	0	0	0	0	0	1	
20			1	0	0	0	1	0	0	1	0	0	0	0	1	
21			1	0	0	0	1	0	0	0	1	0	0	0	1	
22			1	0	0	0	1	0	1	0	1	0	0	0	1	
23			1	0	1	0	0	1	0	0	0	1	0	0	1	
24			1	0	1	0	0	0	0	1	0	1	0	0	1	
25			1	0	1	0	0	0	0	0	0	0	1	0	1	
26			1	0	1	0	0	0	1	0	0	0	1	0	1	
27			1	0	1	0	1	0	0	1	0	0	1	0	1	
28			1	0	1	0	1	0	0	0	1	0	1	0	1	
29			1	0	1	0	1	0	1	0	1	0	1	0	1	
30	1	0	0	1	0	1	0	1	0	0	0	0	0	0	0	1

TABLE 2. Base 2 Expansions

n					
1	0	0	0	0	1
2	0	0	0	1	0
3	0	0	0	1	1
4	0	0	1	0	0
5	0	0	1	0	1
6	0	0	1	1	0
7	0	0	1	1	1
8	0	1	0	0	0
9	0	1	0	0	1
10	0	1	0	1	0
11	0	1	0	1	1
12	0	1	1	0	0
13	0	1	1	0	1
14	0	1	1	1	0
15	0	1	1	1	1
16	1	0	0	0	0
17	1	0	0	0	1
18	1	0	0	1	0
19	1	0	0	1	1
20	1	0	1	0	0
21	1	0	1	0	1
22	1	0	1	1	0
23	1	0	1	1	1
24	1	1	0	0	0
25	1	1	0	0	1
26	1	1	0	1	0
27	1	1	0	1	1
28	1	1	1	0	0
29	1	1	1	0	1
30	1	1	1	1	0

Definition 1.1: For each integer r, define R_r as follows:

a. $R_0 = \alpha^{-2}$,

b. $R_r = \dfrac{L_{r-1}-1}{\alpha L_r + 1}$ for odd $r > 0$,

c. $R_r = \dfrac{L_{r-1}+1}{\alpha L_r - 1}$ for even $r > 0$,

d. $R_r = \dfrac{L_{-r}}{\alpha L_{-r+1}}$ for $r < 0$,

where $L_0 = 2$, $L_1 = 1$, and $L_{i+1} = L_i + L_{i-1}$ are the Lucas numbers for $i \geq 0$. Alternatively, $L_i = F_{i-1} + F_{i+1}$. It was shown in [6] that $\lim_{n\to\infty} Ratio_0(n) = R_0$. In this paper we show that, for each $r \neq 0$, $\lim_{n\to\infty} Ratio_r(n) = R_r$. Note also the interesting fact that, as $r \to \infty$ and as $r \to -\infty$, R_r approaches the limit R_0. Our strategy is to first establish some recursive patterns along each column (these are established in Lemma 3.5) which will allow us to obtain precise expressions for $Ratio_i(n_k)$ and $Ratio_i(m_k)$, where $\{n_k\}$ and $\{m_k\}$ are two subsequences of the natural numbers. The limits of these subsequences can then easily be obtained from known limit results about Fibonacci and Lucas numbers. We then show that, as $n \to \infty$, members of the full sequence $Ratio_i(n)$ must be caught between these two subsequences and hence the full sequence converges. Our proofs use only combinatorial and algorithmic techniques and do not require any specialized number theory background.

2. DEFINITIONS AND PRELIMINARIES

We use definitions and notation similar to those used in [5] and [6]. In particular, $\ell(n)$ denotes the absolute value of the smallest power of α in the α-expansion of n, and $u(n)$ denotes the largest such power. The following is a restatement of Theorem 1 from [4] in terms of the α-expansion.

Theorem 2.1 (Grabner, Nemes, Petho, Tichy): For $k \geq 1$, we have $\ell(n) = u(n) = 2k$ whenever $L_{2k} \leq n \leq L_{2k+1}$, and we have $\ell(n) = 2k+2$ and $u(n) = 2k+1$ whenever $L_{2k+1} < n < L_{2k+2}$.

The following definitions are from [6].

Definition 2.2: We define V to be the infinite dimensional vector space over $\mathbb{Z}$ given by $V \equiv \{(\ldots, v_{-1}, \underline{v_0}, v_1, v_2, \ldots) : v_i \in \mathbb{Z}\ \forall i$, with at most finitely many v_i nonzero$\}$. For clarity, we underline the zero$^{\text{th}}$ coordinate.

Definition 2.3: Define $\hat{V}$ to be the subset of V consisting of all vectors whose entries are in the set $\{0, 1\}$ and which have no two consecutive ones. We will call the elements of $\hat{V}$ totally reduced vectors. When convenient, we omit trailing and leading zeros, so for example,

$$(\ldots, 0, \ldots, 0, 0, 1, \underline{0}, 1, 0, 1, 0, 0, \ldots, 0, \ldots) = (1, \underline{0}, 1, 0, 1).$$

As in [5], we represent α-expansions by vectors in $\hat{V}$, where a one in the j^{th} coordinate represents α^j.

Definition 2.4: We define the function $\alpha : \mathbb{N} \to \hat{V}$ so that, when the α-expansion of n is $\sum_{i=-\infty}^{\infty} e_i \alpha^i$, $\alpha(n)$ is the vector in $\hat{V}$ with $v_i = e_i$.

It follows from Theorem 2.1 that, if $L_{2k} \leq n \leq L_{2k+1}$, we can write

$$n = \alpha^{-2k} + \sum_{i=-2k+2}^{2k-2} e_i \alpha^i + \alpha^{2k} \text{ so that } \alpha(n) = (1, 0, e_{-2k+2}, e_{-2k+3}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-2}, 0, 1) \quad (1)$$

and, if $L_{2k+1} < n < L_{2k+2}$, then we can write

$$n = \alpha^{-2k-2} + \sum_{i=-2k}^{2k-1} e_i \alpha^i + \alpha^{2k+1} \text{ so that } \alpha(n) = (1, 0, e_{-2k}, e_{-2k+1}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-1}, 0, 1). \quad (2)$$

Definition 2.5: The function $\sigma : V \to \mathbb{N}$ is defined as follows: $\sigma((\ldots, v_{-1}, v_0, v_1, \ldots)) = \sum_{i=-\infty}^{\infty} v_i \alpha^i$.

Thus $\sigma(\alpha(n)) = n$ for all natural numbers n. (Note that the definition of σ in [5] is in terms of Fibonacci numbers, and is not equivalent to the one given here. Specifically, the two functions are only guaranteed to be equal when applied to $\alpha(n)$ where $n \in \mathbb{N}$.) The following definitions are generalizations of definitions in [6]. (The definitions in [6] correspond to the case $i = 0$.)

Definition 2.6: We say that n has property $\mathcal{P}_i$ if α^i appears in the α-expansion of n.

Definition 2.7: For natural numbers n, m:

a. $Ones_i(n, m] = |\{k \in N : n < k \leq m,\ k \text{ has property } \mathcal{P}_i\}|$;

b. $Zeros_i(n, m] = |\{k \in N : n < k \leq m,\ k \text{ does not have property } \mathcal{P}_i\}|$;

c. $Ratio_i(n, m] = \frac{Ones_i(n,m]}{Zeros_i(n,m]}$.

By abuse of notation, we also define $Ratio_i(n) = Ratio_i(0, n]$. We will call a finite sequence of 0s and 1s a *pattern*. We use patterns to describe values of $[\alpha(n)]_i$ for fixed i and a sequence of consecutive natural numbers n. Recall that $[\alpha(n)]_i = 1$ if α^i occurs in the α-expansion of n, and that $[\alpha(n)]_i = 0$ otherwise. Thus, for $n \leq m$, we can define the pattern

$$Pat_i(n, m] = [\alpha(n+1)]_i[\alpha(n+2)]_i \cdots [\alpha(m)]_i.$$

Patterns can be concatenated. We will denote the concatenation operation with the operator +, but will omit it when convenient. So, for example, for $n \leq m \leq p$,

$$Pat_i(n, p] = Pat_i(n, m] + Pat_i(m, p] = Pat_i(n, m]Pat_i(m, p].$$

In addition, we use the notation P/n to denote the *prefix* of a pattern P obtained by deleting the rightmost n digits. So, for example, $11001/2 = 110$. By abuse of notation, if P is a pattern, we define $Ones(P)$ and $Zeros(P)$ to be the number of 1s and the number of 0s, respectively, appearing in the pattern P. We also define $Ratio(P) = Ones(P)/Zeros(P)$. We will be using the following known facts about Fibonacci and Lucas numbers: For any $h > 0$, the sequence F_{2n+h}/F_{2n} is decreasing, the sequence F_{2n+1+h}/F_{2n+1} is increasing, the sequence L_{2n+h}/L_{2n} is increasing, the sequence L_{2n+1+h}/L_{2n+1} is decreasing, and

$$\lim_{n\to\infty}(F_{n+h}/F_n) = \alpha^h, \tag{3}$$

$$\lim_{n\to\infty}(L_{n+h}/L_n) = \alpha^h, \tag{4}$$

$$F_{n+h}L_{n+k} - F_nL_{n+h+k} = (-1)^n F_hL_k, \tag{5}$$

$$F_{n+h}F_{n+k} - F_nF_{n+h+k} = (-1)^n F_hF_k, \tag{6}$$

$$\sum_{i=0}^{h} F_i = F_{h+2} - 1, \tag{7}$$

$$\sum_{i=0}^{h} F_{k+2i} = F_{k+2h+1} - F_{k-1}, \tag{8}$$

$$\alpha^k + \alpha^{k+2} = \alpha L_{k+1} + L_k. \tag{9}$$

Formulas (5) and (6) are from [7], page 177, (19b and 20a). The following Lemma will be used repeatedly.

Lemma 2.8: Let $a, b, c, d \in \mathbb{N}$, and $x, y \in \mathbb{R}$. If $\frac{a}{b} \leq x$ and $\frac{c}{d} \leq y$, then $\frac{a+c}{b+d} \leq \max\{x, y\}$. When each $\leq$ is replaced by $\geq$, the result holds with *max* replaced by *min*.

3. SOME USEFUL RESULTS

In the sequence of α-expansions of the natural numbers, the Lucas numbers play a special role. First, note that

$$\alpha(L_{2k}) = 10^{2k-1}\underline{0}0^{2k-1}1 \quad \text{and} \quad \alpha(L_{2k+1}) = (10)^k\underline{1}(01)^k.$$

(Readers may derive these formulas themselves, or refer to [2].) In Table 1, compare the expansions found between $L_4 = 7$ and $L_5 = 11$ with those found between $L_6 = 18$ and $2L_5 = 22$. The two sequences of expansions are identical if we restrict our attention to powers of α between α^{-3} and α^3. Similar observations can be made, for large enough k, by comparing the expansions of the numbers found between L_{2k-2} and L_{2k-1}, and those between L_{2k} and $2L_{2k-1}$: the expansions are identical for those powers of α between α^{-k} and α^k. It can be proved that this is always the case, using an algorithmic technique presented in [5]. In fact, a full recursive pattern in the sequence of α-expansions can be established. This was shown in [6], and we merely restate the relevant results here. Note that, for $n \ge 4$, $L_n < 2L_{n-1} < L_{n-2} + L_n < L_{n+1}$. Thus, we can partition the α-expansions between L_n and L_{n+1} into three segments: the first from L_n to $2L_{n-1}$, the second from $2L_{n-1}$ to $L_{n-2} + L_n$, and the third from $L_{n-2} + L_n$ to L_{n+1}. As partly indicated above (for even n), the sequence of α-expansions between L_n and $2L_{n-1}$ is similar to that between L_{n-2} and L_{n-1}. In addition, the sequence of α-expansions between $2L_{n-1}$ and $L_{n-2} + L_n$ is similar to that between L_{n-3} and L_{n-2}, and the sequence of α-expansions between $L_{n-2} + L_n$ and L_{n+1} is again similar to that between L_{n-2} and L_{n-1}. The exact ways in which the sequences are similar (or dissimilar) vary for each of the three segments, and also vary depending on whether n is even or odd. The full result is expressed in the following propositions, and was proved in Lemma 3.8 of [6].

Proposition 3.1: Let $k \ge 2$. If $0 < m < L_{2k-2}$ and $\alpha(L_{2k-1} + m) = (1, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-3}, 0, 1)$, then:

a. $e_{-(2k-2)} = 0$.

b. $\alpha(L_{2k+1} + m) = (1, 0, 0, 1, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-3}, 0, 0, 0, 1)$.

c. $\alpha(L_{2k-1} + L_{2k+1} + m) = (1, 0, 0, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-3}, 0, 1, 0, 1)$.

d. $\alpha(2L_{2k+1} + m) = (1, 0, 1, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-3}, 0, 1, 0, 0, 1)$.

Proposition 3.2: Let $k \ge 2$. If $0 \le m \le L_{2k-1}$ and $\alpha(L_{2k} + m) = (1, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-2}, 0, 1)$, then:

a. $\alpha(L_{2k+2} + m) = (1, 0, 0, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-2}, 0, 0, 0, 1)$.

b. $\alpha(L_{2k} + L_{2k+2} + m) = (1, 0, 1, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-2}, 0, 1, 0, 1)$.

c. $\alpha(2L_{2k+2} + m) = (1, 0, 0, 1, 0, 0, e_{-(2k-2)}, \ldots, e_{-1}, \underline{e_0}, e_1, \ldots, e_{2k-2}, 0, 1, 0, 0, 1)$.

From Propositions 3.1 and 3.2, the following may be deduced.

Corollary 3.3: Let $k \ge 2$.

a. For $L_{2k+1} < n < 2L_{2k}$, $\alpha(n)$ begins with 10010 and ends in 0001.

b. For $L_{2k-1} + L_{2k+1} < n < L_{2k+2}$, $\alpha(n)$ begins with 10000 and ends in 0101.

c. For $2L_{2k+1} = L_{2k-1} + L_{2k+2} < n < L_{2k} + L_{2k+2}$, $\alpha(n)$ begins with 10100 and ends in 01001.

d. For $L_{2k+2} \le n \le 2L_{2k+1}$, $\alpha(n)$ begins with 1000 and ends in 0001.

e. For $L_{2k} + L_{2k+2} \le n \le L_{2k+3}$, $\alpha(n)$ begins with 1010 and ends in 0101.

f. For $2L_{2k+2} = L_{2k} + L_{2k+3} \le n \le L_{2k+1} + L_{2k+3}$, $\alpha(n)$ begins with 100100 and ends in 01001.

Definition 3.4: For $k \geq 1$, $P_r^k = Pat_r(L_{k-1}, L_k]$.

From the above propositions, recursive formulas for P_r^k easily follow.

Lemma 3.5: Let $k \geq 2$.

a. If $-2k+3 \leq r \leq 2k-2$, then $P_r^{2k+2} = P_r^{2k} P_r^{2k-1} P_r^{2k}$.

b. If $-2k+4 \leq r \leq 2k-3$, then $P_r^{2k+1} = P_r^{2k-1} P_r^{2k-2} P_r^{2k-1}$.

Proof: Fix k and r as above and define the following maps:

$$f_1 : [L_{2k-1}+1, L_{2k}-1] \to [L_{2k+1}+1, 2L_{2k}-1], \qquad f_1(x) = x + L_{2k};$$
$$f_2 : [L_{2k-2}, L_{2k-1}] \to [2L_{2k}, L_{2k-1}+L_{2k+1}], \qquad f_2(x) = x + L_{2k+1};$$
$$f_3 : [L_{2k-1}+1, L_{2k}-1] \to [L_{2k-1}+L_{2k+1}+1, L_{2k+2}-1], \quad f_3(x) = x + L_{2k+1}.$$

Clearly, these maps are one-to-one and onto. Moreover, by Propositions 3.1 and 3.2, $[\alpha(x)]_r = [\alpha(f_i(x))]_r$ for any x in the domain of f_i, if $-2k+3 \leq r \leq 2k-2$. It follows that

$$Pat_r(L_{2k-1}, L_{2k}-1] = Pat_r(L_{2k+1}, 2L_{2k}-1],$$
$$Pat_r(L_{2k-2}-1, L_{2k-1}] = Pat_r(2L_{2k}-1, L_{2k-1}+L_{2k+1}],$$
$$Pat_r(L_{2k-1}, L_{2k}-1] = Pat_r(L_{2k-1}+L_{2k+1}, L_{2k+2}-1].$$

Then

$$Pat_r(L_{2k-1}, L_{2k}-1] + Pat_r(L_{2k-2}-1, L_{2k-1}] + Pat_r(L_{2k-1}, L_{2k}-1]$$
$$= Pat_r(L_{2k+1}, 2L_{2k}-1] + Pat_r(2L_{2k}-1, L_{2k-1}+L_{2k+1}] + Pat_r(L_{2k-1}+L_{2k+1}, L_{2k+2}-1].$$

Using the fact that $[\alpha(L_{2k})]_r = 0$ for every k, this simplifies to $P_r^{2k} P_r^{2k-1} P_r^{2k} = P_r^{2k+2}$. Similarly, we define the following maps:

$$g_1 : [L_{2k-2}, L_{2k-1}] \to [L_{2k}, 2L_{2k-1}], \qquad g_1(x) = x + L_{2k-1};$$
$$g_2 : [L_{2k-3}+1, L_{2k-2}-1] \to [2L_{2k-1}+1, L_{2k-2}+L_{2k}-1], \quad g_2(x) = x + L_{2k};$$
$$g_3 : [L_{2k-2}, L_{2k-1}] \to [L_{2k-2}+L_{2k}, L_{2k+1}], \qquad g_3(x) = x + L_{2k}.$$

Again by Propositions 3.1 and 3.2, these maps are bijections which leave the r^{th} term of the α-expansion of x fixed for $-2k+4 \leq r \leq 2k-3$. So, by concatenating the domains and ranges as above, we again obtain $P_r^{2k-1} P_r^{2k-2} P_r^{2k-1} = P_r^{2k+1}$. □

4. SOME SPECIAL SUBSEQUENCES OF *Ratio*$_r(n)$

Here we show that, for each r, there exist two subsequences of $Ratio_r(n)$ that converge to R_r. These subsequences are related to the odd and even Lucas numbers. One is increasing and the other is decreasing. In Section 5 we show that the sequence $Ratio_r(n)$ is trapped between these two monotone convergent subsequences, and therefore $Ratio_r(n)$ must also converge to R_r.

4.1 Positive Powers of α

We consider even and odd powers separately. For even powers, let $r = 2l$ where $l \geq 1$; for odd powers, let $r = 2l+1$ where $l \geq 0$. Using the recursive formulas derived in the previous section, it is straightforward to obtain closed formulas for $Ones(P_r^k)$.

Lemma 4.1: For $k \geq 2$,

$$Ones(P_{2l}^k) = \begin{cases} 0, & k < 2l, \\ 1, & k = 2l, \\ L_{2l-1}, & k = 2l+1, \\ (L_{2l-1}+1)F_{k-2l-2}, & k \geq 2l+2; \end{cases} \quad Ones(P_{2l+1}^k) = \begin{cases} 0, & k \leq 2l+1, \\ L_{2l}-1, & k = 2l+2, \\ (L_{2l}-1)F_{k-2l-3}, & k \geq 2l+3. \end{cases}$$

Proof: The proof is by induction on k. The base cases are somewhat numerous but straightforward. We use Theorem 2.1 and Corollary 3.3 to compute the entries of the middle two columns of the following table, then compute the last column by simple counting.

	$L_{k-1} < n < L_k$	$n = L_k$	$Ones(P^k)$
$k < 2l$	$u(n) = k-1$ $[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 0$	$u(n) = k-1$ or k $[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l}^k) = 0$ $Ones(P_{2l+1}^k) = 0$
$k = 2l$	$u(n) = 2l-1$ $[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 0$	$u(n) = 2l$ $[\alpha(n)]_{2l} = 1$ $[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l}^k) = 1$ $Ones(P_{2l+1}^k) = 0$
$k = 2l+1$	$u(n) = 2l$ $[\alpha(n)]_{2l} = 1$ $[\alpha(n)]_{2l+1} = 0$	$u(n) = 2l$ $[\alpha(n)]_{2l} = 1$ $[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l}^k) = L_k - L_{k-1} = L_{2l-1}$ $Ones(P_{2l+1}^k) = 0$
$k = 2l+2$	$u(n) = 2l+1$ $[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 1$	$u(n) = 2l+2$ $[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l}^k) = 0$ $Ones(P_{2l+1}^k) = L_k - L_{k-1} - 1 = L_{2l-1}$

If $k = 2l+3$ and $L_{2l+2} < n \leq L_{2l+3}$, then $u(n) = 2l+2$ again by Theorem 2.1. Corollary 3.3 again helps us to complete the following table:

$L_{2l+2} < n \leq 2L_{2l+1}$	$2L_{2l+1} < n < L_{2l} + L_{2l+2}$	$L_{2l} + L_{2l+2} \leq n \leq L_{2l+3}$	$Ones(P^{2l+3})$
$[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 0$	$[\alpha(n)]_{2l} = 0$ $[\alpha(n)]_{2l+1} = 0$	$[\alpha(n)]_{2l} = 1$ $[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l}^{2l+3}) = L_{2l+3} - (L_{2l} + L_{2l+2}) + 1 = L_{2l-1} + 1$ $Ones(P_{2l+1}^{2l+3}) = 0$

If $k = 2l+4$, then $u(n) = 2l+3$ for $L_{2l+3} < n < L_{2l+4}$ and $u(n) = 2l+4$ for $n = L_{2l+4}$. We again invoke Corollary 3.3 to complete the table:

$L_{2l+3} < n < 2L_{2l+2}$	$2L_{2l+2} \leq n$ $\leq L_{2l+1} + L_{2l+3}$	$L_{2l+1} + L_{2l+3}$ $< n < L_{2l+4}$	$n = L_{2l+4}$	$Ones(P^{2l+4})$
$[\alpha(n)]_{2l} = 0$	$[\alpha(n)]_{2l} = 1$	$[\alpha(n)]_{2l} = 0$	$[\alpha(n)]_{2l} = 0$	$Ones(P_{2l}^{2l+4})$ $= L_{2l+1} + L_{2l+3} - 2L_{2l+2} + 1 = L_{2l-1} + 1$
$[\alpha(n)]_{2l+1} = 0$	$[\alpha(n)]_{2l+1} = 0$	$[\alpha(n)]_{2l+1} = 1$	$[\alpha(n)]_{2l+1} = 0$	$Ones(P_{2l+1}^{2l+4})$ $= L_{2l+4} - L_{2l+1} - L_{2l+3} - 1 = L_{2l} - 1$

For the inductive step, assume that $k \geq 2l+5$. By Lemma 3.5,

$$\begin{aligned} Ones(P_{2l}^k) &= 2Ones(P_{2l}^{k-2}) + Ones(P_{2l}^{k-3}) \\ &= 2(L_{2l-1}+1)F_{k-2l-4} + (L_{2l-1}+1)F_{k-2l-5} = (L_{2l-1}+1)F_{k-2l-2}. \end{aligned}$$

Similarly,

$$\begin{aligned} Ones(P_{2l+1}^k) &= 2Ones(P_{2l+1}^{k-2}) + Ones(P_{2l+1}^{k-3}) \\ &= 2(L_{2l}-1)F_{k-2l-5} + (L_{2l}-1)F_{k-2l-6} = (L_{2l}-1)F_{k-2l-3}. \quad \square \end{aligned}$$

Now that we have formulas for $Ones(P_r^k)$ for positive r, closed formulas for $Ratio_r(L_n)$ for all Lucas numbers L_n may be obtained by straightforward calculations. However, the subsequences of the $Ratio_r(m)$ sequence in which we are interested happen to occur not at the Lucas numbers themselves but at points close to the Lucas numbers. Specifically, we will show that (for positive odd powers of α) the values $Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1)$ form a decreasing subsequence at which local maxima occur; and that the values $Ratio_{2l+1}(L_{2k}-L_{2l})$ form an increasing subsequence at which local minima occur. Similar subsequences occur for even positive powers of α, and for negative powers. To obtain formulas for these ratios, we need to first nail down the patterns occurring between these points and the Lucas numbers that they are close to. This is done in the following Lemma for positive powers of α. The proof, which is omitted, uses induction on k combined with results from Propositions 3.1 and 3.2, as well as Corollary 3.3.

Lemma 4.2: If $k \geq l+1$, then:

a. $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+1}] = 0^{L_{2l+1}+1}$.

b. $Pat_{2l+1}(L_{2k+1}-L_{2l+2}, L_{2k+1}-L_{2l+1}-1] = 1^{L_{2l}-1}$.

c. $Pat_{2l+1}(L_{2k}-L_{2l}, L_{2k}] = 1^{L_{2l}-1}0$.

d. $Pat_{2l+1}(L_{2k+2}-L_{2l+3}-1, L_{2k+2}-L_{2l}] = 0^{2L_{2l+1}+1}$.

e. $Pat_{2l}(L_{2k}-L_{2l}, L_{2k}] = 0^{L_{2l}}$.

f. $Pat_{2l}(L_{2k-1}-L_{2l-1}-1, L_{2k-1}] = 1^{L_{2l-1}+1}$.

The main results of this section are given in Theorems 4.3 and 4.4.

Theorem 4.3: For $l \geq 0$ and large enough k:

a. $Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1)$ decreases to R_{2l+1} as k increases.

b. $Ratio_{2l+1}(L_{2k}-L_{2l})$ increases to R_{2l+1} as k increases.

Proof: By Lemmas 4.1 and 4.2, if $k \geq l+1$, then using Formula (7),

$$Ones_{2l+1}(0, L_{2k+1}-L_{2l+1}-1] = Ones_{2l+1}(0, L_{2k+1}] = \sum_{j=1}^{2k+1} Ones(P_{2l+1}^j)$$
$$= L_{2l}-1+\sum_{j=2l+3}^{2k+1}(L_{2l}-1)F_{j-2l-3} = L_{2l}-1+(L_{2l}-1)(F_{2k-2l}-1) = (L_{2l}-1)F_{2k-2l}.$$

It follows that

$$Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1) = \frac{(L_{2l}-1)F_{2k-2l}}{L_{2k+1}-L_{2l+1}-1-(L_{2l}-1)F_{2k-2l}} = \frac{L_{2l}-1}{\dfrac{L_{2k+1}-L_{2l+1}-1}{F_{2k-2l}}-(L_{2l}-1)}.$$

Part (a) follows from the fact that $(L_{2k+1}-L_{2l+1}-1)/F_{2k-2l}$ is increasing for large enough k (which can be deduced from Formula (5)) and has limit $\alpha^{2l+2}+\alpha^{2l} = \alpha L_{2l+1}+L_{2l}$ (from Formulas (3) and (9)). Similarly, if $k \geq l+2$, then

$$Ones_{2l+1}(0, L_{2k}-L_{2l}] = Ones_{2l+1}(0, L_{2k}]-Ones_{2l+1}(L_{2k}-L_{2l}, L_{2k}]$$
$$= \sum_{j=1}^{2k} Ones(P_{2l+1}^j)-(L_{2l}-1) = (L_{2l}-1)\sum_{j=2l+3}^{2k} F_{j-2l-3} = (L_{2l}-1)(F_{2k-2l-1}-1).$$

It follows that

$$Ratio_{2l+1}(L_{2k}-L_{2l})=\frac{(L_{2l}-1)(F_{2k-2l-1}-1)}{L_{2k}-L_{2l}-(L_{2l}-1)(F_{2k-2l-1}-1)}=\frac{L_{2l}-1}{\dfrac{L_{2k}-L_{2l}}{F_{2k-2l-1}-1}-(L_{2l}-1)}.$$

Part (b) follows from the fact that $\frac{L_{2k}-L_{2l}}{F_{2k-2l-1}-1}$ is decreasing (for large enough k, again by Formula (5)) and has limit $\alpha^{2l+2}+\alpha^{2l}$. □

Theorem 4.4: For $l \geq 1$ and large enough k:

a. $Ratio_{2l}(L_{2k}-L_{2l})$ decreases to R_{2l} as k increases.

b. $Ratio_{2l}(L_{2k+1}-L_{2l-1}-1)$ increases to R_{2l} as k increases.

The proof of this theorem is omitted as it is similar to the proof of Theorem 4.3.

4.2 Negative Powers of α

We state here the results for the subsequences of $Ratio_r(n)$ where $r<0$. The proofs are completely analogous to those from the previous subsection.

Lemma 4.5: For $k \geq 1$:

$$Ones(P_{-2l}^k)=\begin{cases}0, & k<2l,\\ L_{2l-2}, & k=2l,\\ L_{2l-1}, & k=2l+1,\\ 0, & k=2l+2,\\ L_{2l}(F_{k-2l-3}+1), & k \text{ odd}, \\ & k\geq 2l+3,\\ L_{2l}(F_{k-2l-3}-1), & k \text{ even},\\ & k \geq 2l+3;\end{cases} \qquad Ones(P_{-(2l+1)}^k)=\begin{cases}0, & k\leq 2l+3,\\ L_{2l+1}(F_{k-2l-4}-1), & k \text{ odd},\\ & k\geq 2l+4,\\ L_{2l+1}(F_{k-2l-4}+1), & k \text{ even},\\ & k\geq 2l+4.\end{cases}$$

Lemma 4.6: If $k \geq l+1$, then:

a. $Pat_{-(2l+1)}(L_{2k}-L_{2l}, L_{2k}]=0^{L_{2l}}$.

b. $Pat_{-(2l+1)}(L_{2k+2}-L_{2l+2}, L_{2k+2}]=1^{L_{2l+1}}0^{L_{2l}}$.

c. $Pat_{-(2l+1)}(L_{2k+3}-2L_{2l+2}, L_{2k+3}]=0^{2L_{2l+2}}$.

d. $Pat_{-2l}(L_{2k}, L_{2k}+L_{2l-1}]=0^{L_{2l-1}}$.

e. $Pat_{-2l}(L_{2k+1}-L_{2l}, L_{2k+1}]=1^{L_{2l}}$.

f. $Pat_{-2l}(L_{2k+2}-L_{2l+2}, L_{2k+2}]=0^{L_{2l+2}}$.

Theorem 4.7: Let l be a nonnegative integer. For large enough k:

a. If $l \geq 0$, then $Ratio_{-(2l+1)}(L_{2k}-L_{2l})$ decreases to $R_{-(2l+1)}$ as k increases.

b. If $l \geq 0$, then $Ratio_{-(2l+1)}(L_{2k+1})$ increases to $R_{-(2l+1)}$ as k increases.

c. If $l \geq 1$, then $Ratio_{-2l}(L_{2k+1})$ decreases to $R_{-(2l)}$ as k increases.

d. If $l \geq 1$, then $Ratio_{-2l}(L_{2k}+L_{2l-1})$ increases to $R_{-(2l)}$ as k increases.

5. THE MAIN RESULT

In this section we show that, as $n \to \infty$, members of the full sequence $Ratio_r(n)$ must be caught between the two subsequences examined in the previous section. In order to do this, we bound the ratios of prefixes of patterns originating at members of the subsequences, and then use Lemma 2.8.

5.1 The Case $r > 0$ Odd

We start by examining in more detail the patterns appearing between Lucas numbers. The base cases are taken care of in the following corollary.

Corollary 5.1: For $l \geq 0$:

a. $Pat_{2l+1}(L_{2l+2}, L_{2l+3}] = 0^{L_{2l+1}}$.

b. $Pat_{2l+1}(L_{2l+3}, L_{2l+4}] = 0^{L_{2l+1}}1^{L_{2l}-1}0$.

c. $Pat_{2l+1}(L_{2l+4}, L_{2l+5}] = 0^{L_{2l+1}}1^{L_{2l}-1}0^{L_{2l+1}+1}$.

d. $Pat_{2l+1}(L_{2l+5}, L_{2l+6}] = 0^{L_{2l+1}}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0$.

e. $Pat_{2l+1}(L_{2l+6}, L_{2l+7}] = 0^{L_{2l+1}}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}$.

Proof: Parts (a)-(c) follow from Corollary 3.3 and Theorem 2.1. Parts (d) and (e) follow from (a)-(c) using Lemma 3.5. □

Lemma 5.2: For $k \geq l+2$:

a. $Ratio(P_{2l+1}^{2k-1}) \leq R_{2l+1}$.

b. $Ratio(P_{2l+1}^{2k-1}P_{2l+1}^{2k}) \leq R_{2l+1}$.

c. $Ratio(P_{2l+1}^{2k}) \geq R_{2l+1}$.

d. $Ratio(P_{2l+1}^{2k}P_{2l+1}^{2k+1}) \geq R_{2l+1}$.

Proof: By Lemma 4.1, if $k \geq l+2$, then

$$Ratio(P_{2l+1}^{2k-1}) = \frac{(L_{2l}-1)F_{2k-2l-4}}{L_{2k-3}-(L_{2l}-1)F_{2k-2l-4}} = \frac{L_{2l}-1}{\frac{L_{2k-3}}{F_{2k-2l-4}}-(L_{2l}-1)}.$$

Since $\frac{L_{2k-3}}{F_{2k-2l-4}}$ is decreasing with limit $\alpha^{2l+2}+\alpha^{2l}$,

$$Ratio(P_{2l+1}^{2k-1}) \leq \frac{L_{2l}-1}{\alpha^{2l+2}+\alpha^{2l}-L_{2l}+1} = R_{2l+1},$$

which proves (a). We also have

$$\begin{aligned} Ones(P_{2l+1}^{2k-1}P_{2l+1}^{2k}) &= Ones(P_{2l+1}^{2k-1}) + Ones(P_{2l+1}^{2k}) \\ &= (L_{2l}-1)F_{2k-2l-4} + (L_{2l}-1)F_{2k-2l-3} = (L_{2l}-1)F_{2k-2l-2}, \end{aligned}$$

and hence, by part (a),

$$Ratio(P_{2l+1}^{2k-1}P_{2l+1}^{2k}) = \frac{(L_{2l}-1)F_{2k-2l-2}}{L_{2k-1}-(L_{2l}-1)F_{2k-2l-2}} = Ratio(P_{2l+1}^{2k+1}) \leq R_{2l+1}.$$

Similarly,

$$Ratio(P_{2l+1}^{2k}) = \frac{(L_{2l}-1)F_{2k-2l-3}}{L_{2k-2}-(L_{2l}-1)F_{2k-2l-3}} = \frac{L_{2l}-1}{\frac{L_{2k-2}}{F_{2k-2l-3}}-(L_{2l}-1)}.$$

Since $\frac{L_{2k-2}}{F_{2k-2l-3}}$ is increasing with limit $\alpha^{2l+2}+\alpha^{2l}$,

$$Ratio(P_{2l+1}^{2k}) \geq \frac{L_{2l}-1}{\alpha^{2l+2}+\alpha^{2l}-L_{2l}+1} = R_{2l+1},$$

which proves (c). For part (d), the reader may check that

$$Ratio(P_{2l+1}^{2k}P_{2l+1}^{2k+1}) = Ratio(P_{2l+1}^{2k+2}) \geq R_{2l+1}$$

using part (c). □

We intend to show that, for $L_{2k+1}-L_{2l+1}-1<n<L_{2k+3}-L_{2l+1}-1$, $\limsup_{n\to\infty} Ratio_{2l+1}(n) \leq R_{2l+1}$. By Theorem 4.3 and Lemma 2.8, it is sufficient to show that, if P is any prefix of $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+1}-2]$, then $Ratio(P) \leq R_{2l+1}$. However, this last statement is not true for the largest prefixes unless k is sufficiently large. We therefore first prove a partial result applicable to prefixes P which do not include the tail end of ones found in $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+1}-2]$.

Lemma 5.3: For $l \geq 0$ and $k \geq l+1$, if P is any prefix of $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+2}]$, then $Ratio(P) \leq R_{2l+1}$.

Proof: We use repeatedly the fact that, by Lemma 2.8, the pattern obtained by concatenating two patterns whose ratios are $\leq R_{2l+1}$ also has ratio $\leq R_{2l+1}$. If $k=l+1$, then by Lemma 4.2 and Corollary 5.1,

$$\begin{aligned} &Pat_{2l+1}(L_{2l+3}-L_{2l+1}-1, L_{2l+5}-L_{2l+2}] \\ &= Pat_{2l+1}(L_{2l+3}-L_{2l+1}-1, L_{2l+3}] + P_{2l+1}^{2l+4}P_{2l+1}^{2l+5} / L_{2l+2} \\ &= 0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}. \end{aligned}$$

The prefix yielding the highest ratio is $P = 0^{2L_{2l+1}+1}1^{L_{2l}-1}$ so that

$$Ratio(P) = \frac{L_{2l}-1}{2L_{2l+1}+1} \leq \frac{L_{2l}-1}{\alpha L_{2l+1}+1} = R_{2l+1}.$$

If $k=l+2$, the pattern in question is

$$\begin{aligned} &Pat_{2l+1}(L_{2l+5}-L_{2l+1}-1, L_{2l+7}-L_{2l+2}] \\ &= Pat_{2l+1}(L_{2l+5}-L_{2l+1}-1, L_{2l+5}] + P_{2l+1}^{2l+6}P_{2l+1}^{2l+7} / L_{2l+2} \\ &= 0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1} \end{aligned}$$

by Lemma 4.2 and Corollary 5.1. We need only consider the prefixes ending in $1^{L_{2l}-1}$, since these yield the highest ratios. We have:

a. $Ratio(0^{2L_{2l+1}+1}1^{L_{2l}-1}) = \dfrac{L_{2l}-1}{2L_{2l+1}+1} < \dfrac{L_{2l}-1}{\alpha L_{2l+1}+1}.$

b. $Ratio(0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}) = \dfrac{2L_{2l}-2}{4L_{2l+1}+2} = \dfrac{L_{2l}-1}{2L_{2l+1}+1} < \dfrac{L_{2l}-1}{\alpha L_{2l+1}+1}.$

c. $Ratio(0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}1^{L_{2l}-1}) = \dfrac{3L_{2l}-3}{5L_{2l+1}+3} = \dfrac{L_{2l}-1}{(5/3)L_{2l+1}+1} < \dfrac{L_{2l}-1}{\alpha L_{2l+1}+1}.$

d. $Ratio(0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}0^{L_{2l+1}+1}1^{L_{2l}-1}0^{2L_{2l+1}+1}1^{L_{2l}-1}) = \dfrac{4L_{2l}-4}{7L_{2l+1}+4}$

$$= \frac{L_{2l}-1}{(7/4)L_{2l+1}+1} < \frac{L_{2l}-1}{\alpha L_{2l+1}+1}.$$

For the inductive step, assume $k \geq l+3$. By Lemmas 3.5 and 4.2,

$$Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1,\, L_{2k+3}-L_{2l+2}]$$
$$= 0^{L_{2l+1}+1}P_{2l+1}^{2k+2}P_{2l+1}^{2k+3} / L_{2l+2} = 0^{L_{2l+1}+1}P_{2l+1}^{2k}P_{2l+1}^{2k-1}P_{2l+1}^{2k}P_{2l+1}^{2k+1}P_{2l+1}^{2k}P_{2l+1}^{2k+1} / L_{2l+2}.$$

Suppose P is a prefix of this pattern.

Case 1: If P is a prefix of $0^{L_{2l+1}+1}P_{2l+1}^{2k}$, the result follows by the induction hypothesis.

Case 2: $P = 0^{L_{2l+1}+1}P_{2l+1}^{2k}Q = 0^{L_{2l+1}+1}P_{2l+1}^{2k-2}P_{2l+1}^{2k-3}P_{2l+1}^{2k-2}Q$, where Q is a prefix of P_{2l+1}^{2k-1}/L_{2l+2}. By the induction hypothesis, $Ratio(0^{L_{2l+1}+1}P_{2l+1}^{2k-2}Q) \leq R_{2l+1}$, and $Ratio(P_{2l+1}^{2k-2}P_{2l+1}^{2k-3}) \leq R_{2l+1}$ by Lemma 5.2. Hence, $Ratio(P) \leq R_{2l+1}$.

Case 3: $P = 0^{L_{2l+1}+1}P_{2l+1}^{2k}(P_{2l+1}^{2k-1} / L_{2l+2})Q$, where Q is a prefix of $Pat_{2l+1}(L_{2k-1}-L_{2l+2}, L_{2k-1}] = 1^{L_{2l}-1}0^{L_{2l+1}+1}$ by Lemma 4.2. The prefix yielding the largest ratio is

$$P = 0^{L_{2l+1}+1}P_{2l+1}^{2k}(P_{2l+1}^{2k-1} / L_{2l+2})1^{L_{2l}-1}.$$

But this is a permutation of $P' = P_{2l+1}^{2k}(P_{2l+1}^{2k-1} / L_{2l+2})1^{L_{2l}-1}0^{L_{2l+1}+1} = P_{2l+1}^{2k}P_{2l+1}^{2k-1}$, so that $Ratio(P) = Ratio(P') \leq R_{2l+1}$ by Lemma 5.2.

Case 4: $P = 0^{L_{2l+1}+1}P_{2l+1}^{2k}P_{2l+1}^{2k-1}Q$, where Q is a prefix of $P_{2l+1}^{2k}P_{2l+1}^{2k+1} / L_{2l+2}$. By the induction hypothesis, $Ratio(0^{L_{2l+1}+1}Q) \leq R_{2l+1}$. By Lemma 5.2, $Ratio(P_{2l+1}^{2k}P_{2l+1}^{2k-1}) \leq R_{2l+1}$, so $Ratio(P) \leq R_{2l+1}$.

Case 5: $P = 0^{L_{2l+1}+1}P_{2l+1}^{2k}P_{2l+1}^{2k-1}P_{2l+1}^{2k}(P_{2l+1}^{2k+1} / L_{2l+2})Q = 0^{L_{2l+1}+1}P_{2l+1}^{2k+2}(P_{2l+1}^{2k+1} / L_{2l+2})Q$, where Q is a prefix of $Pat_{2l+1}(L_{2k+1}-L_{2l+2}, L_{2k+1}] = 1^{L_{2l}-1}0^{L_{2l+1}+1}$. As in Case 4, the prefix yielding the highest ratio is

$$P = 0^{L_{2l+1}+1}P_{2l+1}^{2k+2}(P_{2l+1}^{2k+1} / L_{2l+2})1^{L_{2l}-1}$$

which is a permutation of $P' = P_{2l+1}^{2k+1}P_{2l+1}^{2k+2}$, so that $Ratio(P) = Ratio(P') \leq R_{2l+1}$ by Lemma 5.2.

Case 6: $P = 0^{L_{2l+1}+1}P_{2l+1}^{2k+2}P_{2l+1}^{2k+1}Q$, where Q is a prefix of $P_{2l+1}^{2k}P_{2l+1}^{2k+1} / L_{2l+2}$. By the induction hypothesis, $Ratio(0^{L_{2l+1}+1}Q) \leq R_{2l+1}$. By Lemma 5.2, $Ratio(P_{2l+1}^{2k+2}P_{2l+1}^{2k+1}) \leq R_{2l+1}$; thus, $Ratio(P) \leq R_{2l+1}$. □

The result for all prefixes can now be proved as follows.

Lemma 5.4: For $l \geq 0$, there exists an integer K_l such that, for $k \geq K_l$, if P is any prefix of $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1, L_{2k+3}-L_{2l+1}-2]$, then $Ratio(P) \leq R_{2l+1}$.

Proof: Note that, by Lemma 4.2,

$$Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1,\, L_{2k+3}-L_{2l+1}-2] = Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1,\, L_{2k+3}-L_{2l+2}] + 1^{L_{2l}-2},$$

so, in view of Lemma 5.3, we need only show that, for large enough k,

$$Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1,\ L_{2k+3}-L_{2l+1}-2] \le R_{2l+1}.$$

Now, by Lemmas 4.2 and 4.1,

$$\begin{aligned}&Ones_{2l+1}(L_{2k+1}-L_{2l+1}-1,\ L_{2k+3}-L_{2l+1}-2]\\&=Ones_{2l+1}(L_{2k+1}-L_{2l+1}-1,\ L_{2k+1}]+Ones_{2l+1}(L_{2k+1},\ L_{2k+3}]\\&\quad-Ones_{2l+1}(L_{2k+3}-L_{2l+1}-2,\ L_{2k+3}]=(L_{2l}-1)F_{2k-2l+1}-1.\end{aligned}$$

So

$$\begin{aligned}Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1,\ L_{2k+3}-L_{2l+1}-2]&=\frac{(L_{2l}-1)F_{2k-2l+1}-1}{(L_{2k+2}-1)-((L_{2l}-1)F_{2k-2l+1}-1)}\\&=\frac{1}{\dfrac{L_{2k+2}-1}{(L_{2l}-1)F_{2k-2l+1}-1}}\le\frac{1}{\dfrac{\alpha^{2l}+\alpha^{2l+2}}{L_{2l}-1}-1}\end{aligned}$$

since $\frac{L_{2k+2}-1}{(L_{2l}-1)F_{2k-2l+1}-1}$ is decreasing for k larger than some K_l, by Formula (5), with limit $\frac{\alpha^{2l}+\alpha^{2l+2}}{L_{2l}-1}$.
Now

$$\frac{1}{\dfrac{\alpha^{2l}+\alpha^{2l+2}}{L_{2l}-1}-1}=\frac{L_{2l}-1}{\alpha^{2l}+\alpha^{2l+2}-L_{2l}+1}=\frac{L_{2l}-1}{\alpha L_{2l+1}+1}=R_{2l+1}.$$

This proves the lemma. □

The next step consists of showing that, for $L_{2k}-L_{2l}<n<L_{2k+2}-L_{2l}$,

$$\liminf_{n\to\infty} Ratio_{2l+1}(n)\ge R_{2l+1}.$$

Again, it is sufficient to consider proper prefixes of $Pat_{2l+1}(L_{2k}-L_{2l},\ L_{2k+2}-L_{2l}]$. The results and proofs are analogous to the ones just presented. We present only the statements of the results.

Lemma 5.5: For $l\ge 0$ and $k\ge l+1$, if P is any prefix of $Pat_{2l+1}(L_{2k}-L_{2l},\ L_{2k+2}-L_{2l+3}-1]$, then $Ratio(P)\ge R_{2l+1}$.

Lemma 5.6: For $l\ge 0$, there exists an integer $\widetilde{K}_l$ such that, if $k\ge\widetilde{K}_l$ and P is any prefix of $Pat_{2l+1}(L_{2k}-L_{2l},\ L_{2k+2}-L_{2l}-1]$, then $Ratio(P)\ge R_{2l+1}$.

We can now state the final result for positive odd powers of α.

Theorem 5.7: For any $l\ge 0$, $\lim_{n\to\infty} Ratio_{2l+1}(n)=R_{2l+1}$.

Proof: If $L_{2k+1}-L_{2l+1}-1<n\le L_{2k+3}-L_{2l+1}-2$, then

$$Pat_{2l+1}(0,n]=Pat_{2l+1}(0,\ L_{2k+1}-L_{2l+1}-1]+P,$$

where P is a prefix of $Pat_{2l+1}(L_{2k+1}-L_{2l+1}-1,\ L_{2k+3}-L_{2l+1}-2]$. If k is large enough, then by Lemma 5.4, $Ratio(P)\le R_{2l+1}$, and by Theorem 4.3, $Ratio(L_{2k+1}-L_{2l+1}-1)$ decreases to the limit R_{2l+1}. So

$$Ratio_{2l+1}(n)\le\max\{Ratio_{2l+1}(L_{2k+1}-L_{2l+1}-1),\ Ratio(P)\}.$$

Letting $n\to\infty$, we obtain

$$\limsup_{n\to\infty} Ratio_{2l+1}(n)\le R_{2l+1}.$$

Similarly, if $L_{2k}-L_{2l}<n\le L_{2k+2}-L_{2l}-1$, then

$$Pat_{2l+1}(0,n]=Pat_{2l+1}(0,L_{2k}-L_{2l}]+P,$$

where P is a prefix of $Pat_{2l+1}(L_{2k}-L_{2l}, L_{2k+2}-L_{2l}-1]$. If k is large enough, then by Lemma 5.6, $Ratio(P)\ge R_{2l+1}$, and by Theorem 4.3, $Ratio(L_{2k}-L_{2l})$ increases to the limit R_{2l+1}. Therefore,

$$Ratio_{2l+1}(n)\ge \min\{Ratio_{2l+1}(L_{2k}-L_{2l}), Ratio(P)\}.$$

Now, letting $n\to\infty$, we obtain

$$\liminf_{n\to\infty} Ratio_{2l+1}(n)\ge R_{2l+1}. \quad \square$$

5.2 Other Cases

We state the results for the cases $r>0$ even and $r<0$ without proof. The proofs are very similar to those in the previous subsection.

Lemma 5.8: For $l\ge 1$, there exists an integer K_l such that, for $k\ge K_l$, if P is any prefix of $Pat_{2l}(L_{2k}-L_{2l}, L_{2k+2}-L_{2l}-1]$, then $Ratio(P)\le R_{2l}$.

Lemma 5.9: For $l\ge 1$, there exists an integer $\widetilde{K}_l$ such that, for $k\ge \widetilde{K}_l$, if P is any prefix of $Pat_{2l}(L_{2k+1}-L_{2l-1}-1, L_{2k+3}-L_{2l-1}-2]$, then $Ratio(P)\ge R_{2l}$.

Theorem 4.4 together with Lemmas 5.8 and 5.9 leads to the following theorem.

Theorem 5.10: For any $l\ge 1$, $\lim_{n\to\infty} Ratio_{2l}(n)=R_{2l}$.

Lemma 5.11: For $l\ge 0$, there exists an integer K_l such that, for $k\ge K_l$, if P is any prefix of $Pat_{-(2l+1)}(L_{2k}-L_{2l}, L_{2k+2}-L_{2l}-1]$, then $Ratio(P)\le R_{-(2l+1)}$.

Lemma 5.12: For $l\ge 0$, there exists an integer $\widetilde{K}_l$ such that, for $k\ge \widetilde{K}_l$, if P is any prefix of $Pat_{-(2l+1)}(L_{2k+1}, L_{2k+3}-1]$, then $Ratio(P)\ge R_{-(2l+1)}$.

Theorem 4.7 together with Lemmas 5.11 and 5.12 leads to the following theorem.

Theorem 5.13: For any $l\ge 0$, $\lim_{n\to\infty} Ratio_{-(2l+1)}(n)=R_{-(2l+1)}$.

Lemma 5.14: For $l\ge 1$, there exists an integer K_l such that, for $k\ge K_l$, if P is any prefix of $Pat_{-(2l)}(L_{2k+1}, L_{2k+3}-1]$, then $Ratio(P)\le R_{-(2l)}$.

Lemma 5.15: For $l\ge 1$, there exists an integer $\widetilde{K}_l$ such that, for $k\ge \widetilde{K}_l$, if P is any prefix of $Pat_{-(2l)}(L_{2k}+L_{2l-1}, L_{2k+2}+L_{2l-1}-1]$, then $Ratio(P)\ge R_{-(2l)}$.

Theorem 4.7 together with Lemmas 5.14 and 5.15 leads to the following theorem.

Theorem 5.16: For any $l\ge 1$, $\lim_{n\to\infty} Ratio_{-(2l)}(n)=R_{-(2l)}$.

6. CONCLUSION

We have characterized the frequency of occurrence of α^i in the α-expansions of the positive integers, for both positive and negative powers of α, using a recursive pattern found in these expansions. These results complete the characterization of the frequency of occurrence of the powers of α in the α-expansions of the positive integers, which was started in [6]. Other characteristics, such as the frequency of occurrence of certain specific patterns in the expansions, might be capable of being derived using similar methods.

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