DIVISIBILITY OF THE COEFFICIENTS OF CHEBYSHEV POLYNOMIALS BY PRIMES

B. J. O. Franco
Universidade Federal de Minas Gerais, Caixa Postal 702,
Departamento de Física, Belo Horizonte 30.161-970, MG, Brazil

Antônio Zumpano
Universidade Federal de Minas Gerais, Caixa Postal 702,
Departamento de Matemática, Belo Horizonte 30.123-970, MG, Brazil

(Submitted June 1999-Final Revision December 1999)

1. INTRODUCTION

In this paper we shall be concerned with Chebyshev polynomials of the first kind, defined by (see [2], [7])

$$C_{n+1}(x) = xC_n(x) - C_{n-1}(x) \tag{1.1}$$

with

$$C_0(x) = 2,\ C_1(x) = x. \tag{1.2}$$

Up to $n = 5$, we have

$$\begin{aligned}
&C_0(x) = 2,\\
&C_1(x) = x,\\
&C_2(x) = x^2 - 2,\\
&C_3(x) = x^3 - 3x,\\
&C_4(x) = x^4 - 4x^2 + 2,\\
&C_5(x) = x^5 - 5x^3 + 5x.
\end{aligned}$$

Generalized Chebyshev polynomials of the first and second kind (in the quasiperiodic sense) were obtained by Dotera [5] and by Suzuki and Dotera [12] in a study of a second-order Fibonacci chain. They obtained self-similar polynomials that contain a parameter r which gives the intensity of quasiperiodicity. When $r = 1$ (a periodic crystal), the polynomials coincide with Chebyshev's polynomials whose degrees are the Fibonacci numbers (1, 2, 3, 5, 8, 13, 21, ...). Another work by Clark and Suryanarayan [4] considered Chebyshev polynomials of the second kind associated with quasiperiodic tilings of the plane. Here we consider the divisibility by primes of coefficients of Chebyshev polynomials of the first kind.

In the proofs, we shall make use of binomial coefficients. Divisibility of binomial coefficients by primes has been considered elsewhere (e.g., [1], [6]-[12], [14]-[16]). We also prove that, if the degree of the Chebyshev polynomials is an odd number, then the coefficient of the second-degree term is a perfect square.

2. COEFFICIENTS OF THE CHEBYSHEV POLYNOMIALS

If we consider equation (1.1) with initial condition $C_0(x) = a$, $a \in \mathbb{Z}$, and $C_1(x) = x$, we have the following array for the coefficients of Chebyshev polynomials A_{ij}, $n \geq 1$:

$$
\begin{array}{lllllllll}
C_1(x) & 1 & & & & & & & \\
C_2(x) & 1 & a & & & & & & \\
C_3(x) & 1 & a+1 & & & & & & \\
C_4(x) & 1 & a+2 & a & & & & & \\
C_5(x) & 1 & a+3 & 2a+1 & & & & & \\
C_6(x) & 1 & a+4 & 3a+3 & a & & & & \\
C_7(x) & 1 & a+5 & 4a+6 & 3a+1 & & & & \\
C_8(x) & 1 & a+6 & 5a+10 & 6a+4 & a & & & \\
C_9(x) & 1 & a+7 & 6a+15 & 10a+10 & 4a+1 & & & \\
C_{10}(x) & 1 & a+8 & 7a+21 & \boxed{15a+20} & 10a+5 & a & & \\
C_{11}(x) & 1 & a+9 & 8a+28 & 21a+35 & 20a+15 & 5a+1 & & \\
C_{12}(x) & 1 & a+10 & 9a+36 & 28a+56 & 35a+21 & 15a+6 & a & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
\tag{2.1}
$$

In the array (2.1), if we define a 2×2 matrix as indicated, the sum of elements of the main diagonal gives an element of the next row of the array, that is,

$$A_{ij} = A_{i-2,\,j-1} + A_{i-1,\,j}. \tag{2.2}$$

For example, for the indicated matrix, we have

$$(5a+10)+(10a+10) = 15a+20.$$

Consider the Pascal triangle:

$$
\begin{array}{llllllllll}
1 & & & & & & & & & \\
\cancel{1} & \cancel{1} & & & & & & & & \\
\cancel{1} & \cancel{2} & \cancel{1} & & & & & & & \\
\cancel{1} & \cancel{3} & \cancel{3} & \cancel{1} & & & & & & \\
\cancel{1} & \cancel{4} & \cancel{6} & \cancel{4} & 1 & & C_{11}(x) & & & \\
\cancel{1} & \cancel{5} & \cancel{10} & 10 & \boxed{5} & \boxed{1} & & & & \\
\cancel{1} & \cancel{6} & 15 & \boxed{20} & \boxed{15} & 6 & 1 & & & \\
\cancel{1} & 7 & \boxed{21} & \boxed{35} & 35 & 21 & 7 & 1 & & \\
1 & \boxed{8} & \boxed{28} & 56 & 70 & 56 & 28 & 8 & 1 & \\
\boxed{1} & \boxed{9} & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Diagonals as indicated above are named *ascendant diagonals* of the Pascal triangle. The sum of the elements of ascendant diagonals are the Fibonacci numbers $(1, 2, 3, 5, 8, 13, 21, \ldots)$ (see [13]). Here we are interested in two adjacent ascendant diagonals. If we compare the row of $C_{11}(x)$ of the array (2.1) with the two indicated ascendant diagonals of the Pascal triangle (inside of a box), we see that each element of the array (2.1) is equal to the sum of pairs of elements of the Pascal triangle (inside of boxes) with the first one multiplied by a $(a = 2)$.

3. DIVISIBILITY BY PRIMES OF THE COEFFICIENTS OF CHEBYSHEV POLYNOMIALS OF THE FIRST KIND

We establish the following facts about the coefficients of the Chebyshev polynomials of the first kind.

(a) If $n \in \mathbb{N}$ is a prime number, then n divides all the coefficients of $C_n(x)$ except the first one.
(b) If n is an even number, then the coefficient of the term of power 2 is a perfect square and its square root is $n/2$.
(c) If $n = pq$, where p and q are prime numbers ($p > q > 2$), then

$$p \left| \left[C_n(x) - \sum_{\lambda=1}^{q} C_{\lambda p} x^{\lambda p} \right], \right. \ \lambda \text{ odd.} \tag{3.1}$$

(d) If $n = 2p$, where p is a prime, then

$$p \mid [C_n(x) - x^n \pm 2]. \tag{3.2}$$

4. PROOFS OF THE RESULTS

We will prove "closed formulas" that generate the polynomials we are studying in terms of binomial coefficients. These formulas are useful in order to calculate the coefficients without using a recursive method which is very slow.

Lemma 1: We have the following. For $n \geq 1$, the polynomials $C_n(x)$ with $C_0(x) = a$, $a \in \mathbb{Z}^*$, $C_1(x) = x$, and $C_{n+1}(x) = xC_n(x) - C_{n-1}(x)$ can be expressed by the formulas:

$$C_n(x) = x^n + \sum_{k=2}^{\frac{n+1}{2}} \left[\binom{n-k}{k-2} a + \binom{n-k}{k-1} \right] x^{n-2(k-1)} (-1)^{k-1},$$

if n is odd; and

$$C_n(x) = x^n + \sum_{k=2}^{\frac{n}{2}+1} \left[\binom{n-k}{k-2} a + \binom{n-k}{k-1} \right] x^{n-2(k-1)} (-1)^{k-1},$$

if n is even.

Proof: For $n = 1$ and $n = 2$, it is obvious since $C_1(x) = x$ and $C_2(x) = x^2 - a$. Suppose that the expression is true for all $k \leq n$ and n is even. Let us show that it will also be true for $n+1$. In view of $C_{n+1}(x) = xC_n(x) - C_{n-1}(x)$, we have

$$\begin{aligned}
C_{n+1}(x) &= xx^n + \sum_{k=2}^{\frac{n}{2}+1} \left[\binom{n-k}{k-2} a + \binom{n-k}{k-1} \right] x^{n-2(k-1)+1} (-1)^{k-1} \\
&\quad - x^{n-1} + \sum_{k=2}^{\frac{n}{2}} \left[\binom{n-1-k}{k-2} a + \binom{n-1-k}{k-1} \right] x^{n-1-2(k-1)} (-1)^{k-1} \\
&= x^{n+1} + \sum_{k=2}^{\frac{n}{2}+1} \left[\binom{n-k}{k-2} a + \binom{n-k}{k-1} \right] x^{n-2(k-1)+1} (-1)^{k-1} \\
&\quad - x^{n-1} + \sum_{k=3}^{\frac{n}{2}} \left[\binom{n-k}{k-3} a + \binom{n-1-k}{k-1} \right] x^{n-2(k-1)+1} (-1)^{k-1} \\
&= x^{n+1} - x^{n-1} - [a + (n-2)x^{n-1}] + \sum_{k=3}^{\frac{n}{2}+1} [Aa + B] x^{n-2(k-1)+1} (-1)^{k-1},
\end{aligned}$$

where

$$A = \left[\binom{n-k}{k-2} - \binom{n-k}{k-3}\right] \quad \text{and} \quad B = \left[\binom{n-k}{k-1} - \binom{n-k}{k-2}\right].$$

So

$$C_{n+1}(x) = x^{n+1} + \sum_{k=3}^{\frac{(n+1)+1}{2}} \left[\binom{(n+1)-k}{k-2}a + \binom{(n+1)-k}{k-1}\right] x^{n-2(k-1)}(-1)^{k-1},$$

by the Stiefel formula.

Hence, the expression is true for $n+1$. In the same way, we prove for n odd. Therefore, by induction, we obtain the result. □

Note: In the above lemma, it is clear that we have considered the convention $\binom{n-k}{k-1} = 0$ when $k = \frac{n}{2}+1$.

We now state a theorem about divisibility of the coefficients of the polynomials

$$C_n(x) = \sum_{i=0}^{n} C_n^i x^i,$$

where $C_0(x) = 2$, $C_1(x) = x$, and $C_{n+1}(x) = xC_n(x) - C_{n-1}(x)$.

By Lemma 1, we see that: for n even, $C_n^i = 0$ for all i odd and $C_n^0 = \pm 2$; for n odd, $C_n^i = 0$ for all i even.

Theorem 1: If $s \mid n$ and $\gcd(s, i) = 1$, then $s \mid C_n^i$.

Proof: By Lemma 1, we have that

$$C_n^i = \binom{n-k}{k-2}2 + \binom{n-k}{k-1} = \frac{n(n-k)!}{(k-1)!(n-2k+2)!}$$

for some k. Since the case when $i = 0$ is trivial, we will deal with $i = 1, 2, \ldots, n$. Thus,

$$C_n^i = \frac{n\binom{n-k}{k-1}}{n-2k+2} = \frac{sr\binom{n-k}{k-1}}{n-2k+2} = \frac{sr\binom{n-k}{k-1}}{i},$$

since $s \mid n$ and $i = n-2k+2$. But, by hypothesis, s and i are relatively prime, and this implies that $s \mid C_n^i$. □

Corollary 1: If $n \in \mathbb{N}$ is prime, then $n \mid C_n^i$, $i = 0, 1, \ldots, n-1$.

Corollary 2: If $n \in \mathbb{N}$ and p is a prime such that $p \mid n$ and $p \nmid i$, then $p \mid C_n^i$ for all $i = 0, 1, \ldots, n$.

Corollary 3: If n is of the form $n = 2p$, where p is a prime, then $p \mid C_n^i$, $i = 1, 2, \ldots, n-1$.

Corollary 4: If n is of the form $n = t^2$, then $t \mid C_n^t$.

Corollary 5: If $n \in \mathbb{N}$ is even, then $C_n^2 = (n/2)^2$.

Proof: Since $C_n^2 = \frac{n}{2}\binom{n-k}{k-1}$, where $k = \frac{n}{2}$, we have that

$$C_n^2 = \frac{n}{2}\binom{\frac{n}{2}}{\frac{n}{2}-1} = \frac{n}{2} \times \frac{n}{2} = \left(\frac{n}{2}\right)^2. \ \square$$

ACKNOWLEDGMENT

We would like to acknowledge the financial support given by CNPq-Brasil and to thank the anonymous referee for many helpful suggestions.

REFERENCES

1. M. Bhaskarman. "A Divisibility Property of Integers Associated with Binomial Coefficients." *Indian J. Math.* **7** (1965):83-84.
2. M. Boscarol. "A Note on Binomial Coefficients and Chebyshev Polynomials." *The Fibonacci Quarterly* **23.2** (1984):166-68.
3. L. Carlitz. "The Number of Binomial Coefficients Divisible by a Fixed Power of a Prime." *Rend. Circ. Palermo*, (2), **16** (1967):299-320.
4. D. Clark & E. R. Suryanarayan. "Chebyshev Matrices and Quasiperiodic Tilings." *Acta Crys* **A51** (1995):684-90.
5. T. Dotera. "Self-Similar Polynomials Obtained from a One-Dimensional Quasiperiodic Model." *Phys. Rev.*, B, **38.1** (1988):11534-42.
6. N. J. Fine. "Binomial Coefficients Modulo a Prime." *Amer. Math. Monthly* **54** (1947):589-92.
7. H. Hsiao. "On Factorization of Chebyshev's Polynomials of the First Kind." *Bull. Inst. Math. Acad. Sinica* **12.1** (1984):89-94.
8. D. E. Knuth & H. S. Wilf. "The Power of a Prime that Divides a Generalized Binomial Coefficient." *J. Reine Angrew Math.* **396** (1989):212-19.
9. C. T. Long. "Pascal's Triangle Modulo p." *The Fibonacci Quarterly* **19.5** (1981):458-63.
10. C. T. Long. "Some Divisibility Properties of Pascal's Triangle." *The Fibonacci Quarterly* **19.3** (1981):257-63.
11. G. J. Simmons. "Some Prime Concerning the Occurrence of Specified Prime Factors in $\binom{m}{r}$." *Amer. Math. Monthly* **77** (1970):510-12.
12. T. Suzuki & T. Dotera. "Dynamical Systems for Quasiperiodic Chains and New Self-Similar Polynomials." *J. Phys.*, A, **26** (1993):6101-13.
13. N. N. Voroboyov. *Fibonacci Numbers*. Boston: D. C. Heath & Co., 1961.
14. W. Webb. "The Number of Binomial Coefficients in Residue Classes Modulo p and p^2." *Colloq. Math.* **60/61** (1990):275-80.
15. W. Webb & K. Davis. "Pascal's Triangle Modulo 4." *The Fibonacci Quarterly* **29.1** (1991): 70-83.
16. W. Webb & K. Davis. "Binomial Coefficient Congruences Modulo Prime Powers." *J. Number Theory* **43.1** (1993):20-23.

AMS Classification Numbers: 05A10, 11A41, 11B65