A TRIBONACCI IDENTITY

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1. INTRODUCTION

We define the generalized Tribonacci sequence $\{V_n\}$ by means of

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}, \tag{1.1}$$

where V_0, V_1, V_2 are arbitrary complex numbers, and r, s, t are arbitrary integers, with $t \neq 0$. The definition is clear for $n \geq 3$; it can be extended to negative subscripts by defining

$$V_{-n} = -\frac{r}{t}V_{-(n-1)} - \frac{s}{t}V_{-(n-2)} + \frac{1}{t}V_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Thus, recurrence (1.1) holds for all integers n.

We will occasionally use the notation

$$V_n = V_n(V_0, V_1, V_2;\ r, s, t) \tag{1.2}$$

to make clear the initial conditions V_0, V_1, and V_2, and the values of r, s, and t. For example, one sequence of particular interest is

$$J_n = V_n(3, r, r^2 + 2s;\ r, s, t). \tag{1.3}$$

That is, $J_0 = 3$, $J_1 = r$, $J_2 = r^2 + 2s$, and $J_n = V_n$ satisfies recurrence (1.1) for all n. It is pointed out in [4] that in some ways the sequence $\{J_n\}$ defined by (1.3) bears the same relation to the sequence $\{V_n\}$ defined by (1.2) as does the Lucas sequence to the Fibonacci sequence. Another useful special case is

$$W_n = V_n(0, W_1, W_2;\ r, s, t), \tag{1.4}$$

where W_1 and W_2 are arbitrary complex numbers.

The main purpose of this paper is to prove the identity

$$V_{n+2m} = J_m V_{n+m} - t^m J_{-m} V_n + t^m V_{n-m}, \tag{1.5}$$

where n and m are arbitrary integers and $\{V_n\}$ and $\{J_n\}$ are defined by (1.2) and (1.3). Formula (1.5) is a special case of a large class of identities discussed in [2]. It is analogous to the well-known

$$F_{n+m} = L_m F_n + (-1)^{m-1} F_{n-m}, \tag{1.6}$$

where $\{F_n\}$ and $\{L_n\}$ are the Fibonacci and Lucas sequences, respectively. The emphasis in the present paper is on the derivation of (1.5) from a general formula in [2] for k^{th}-order recurrences. The writer hopes that this method of proof can eventually be used to prove formulas analogous to (1.6) for other generalizations of $\{F_n\}$, such as the tetrabonacci numbers.

A secondary purpose of this paper is to prove a useful extension of the main result in [2]. This extension, Theorem 2.1 below, is a basic tool in the proof of formula (1.5).

2. PRELIMINARIES

Theorem 2.1: Let $e_1, e_2, \ldots, e_k$ be arbitrary complex numbers with $e_k \neq 0$, and define the sequence $\{a_n\}$ as follows: $a_0, a_1, \ldots, a_{k-1}$ are arbitrary complex numbers and, for all integers n,

$$a_n = e_1 a_{n-1} + e_2 a_{n-2} + \cdots + e_k a_{n-k}. \tag{2.1}$$

Then, for $m \geq 1$ and all integers n,

$$a_{(k-1)m+n} = \sum_{j=1}^{k} (-1)^{j-1} c_{m,\, jm} a_{(k-j-1)m+n}.$$

The numbers $c_{m,\, jm}$ are defined by

$$\prod_{i=0}^{m-1} [1 - e_1(\theta^i x) - e_2(\theta^i x)^2 - \cdots - e_k(\theta^i x)^k] = 1 + \sum_{j=1}^{k} (-1)^j c_{m,\, jm} x^{jm}, \tag{2.2}$$

where θ is a primitive m^{th} root of unity.

Proof: This theorem was proved in [2] for the case $n \geq m \geq 1$; the purpose here is to remove all restrictions on n. We first note that formula (2.1) is extended to negative subscripts $-n$ by defining, for $n = 1, 2, 3, \ldots$,

$$a_{-n} = -\frac{e_{k-1}}{e_k} a_{-(n-1)} - \frac{e_{k-2}}{e_k} a_{-(n-2)} - \cdots - \frac{e_1}{e_k} a_{-(n-k+1)} + \frac{1}{e_k} a_{-(n-k)}.$$

Let h be an integer, and define $b_j = a_{j-h}$. The generating function for $\{b_j\}$ is of the form

$$\frac{f(x)}{1 - e_1 x - \cdots - e_k x^k} = \sum_{j=0}^{\infty} b_j x^j,$$

where $f(x)$ is a polynomial of degree less than k (see [3], p. 230), and it is clear from the proof in [2] for the case $n \geq m$ that

$$b_{(k-1)m+n} = \sum_{j=1}^{k} (-1)^{j-1} c_{m,\, jm} b_{(k-j-1)m+n},$$

with $c_{m,\, jm}$ defined by (2.2). That is,

$$a_{(k-1)m+n-h} = \sum_{j=1}^{k} (-1)^{j-1} c_{m,\, jm} a_{(k-j-1)m+n-h}. \tag{2.3}$$

Since h can be any integer in (2.3), we no longer have the restriction that $n \geq m$, and, in fact, n can be negative. This completes the proof. □

The following lemma is essential for this paper. It states some well-known relationships between the recurrence (1.1) and the roots of the characteristic equation $x^3 - rx^2 - sx - t = 0$ (see [3], pp. 210-15).

Lemma 2.1: Let α_1, α_2, and α_3 be the roots of $x^3 - rx^2 - sx - t = 0$, where r, s, and t are arbitrary integers, $t \neq 0$. Let $\{V_n\}$ and $\{J_n\}$ be defined by (1.1) and (1.3), respectively.

(a) **Case 1:** $\alpha_1 \neq \alpha_2$; $\alpha_1 \neq \alpha_3$; $\alpha_2 \neq \alpha_3$. Then, for all integers n, $V_n = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n$, where A_1, A_2, A_3 are constants determined by the initial conditions V_0, V_1, V_2. Also, $J_n = \alpha_1^n + \alpha_2^n + \alpha_3^n$ and $\alpha_1\alpha_2\alpha_3 = t$.

(b) **Case 2:** $\alpha_1 = \alpha_2; \alpha_1 \neq \alpha_3$. Then, for all integers n, $V_n = (A_1 + A_2 n)\alpha_1^n + A_3\alpha_3^n$, where A_1, A_2, A_3 are constants determined by the initial conditions V_0, V_1, V_2. Also, $J_n = 2\alpha_1^n + \alpha_3^n$ and $\alpha_1^2\alpha_3 = t$.

(c) **Case 3:** $\alpha_1 = \alpha_2 = \alpha_3$. Then, for all integers n, $V_n = (A_1 + A_2 n + A_3 n^2)\alpha_1^n$, where A_1, A_2, A_3 are constants determined by the initial conditions V_0, V_1, V_2. Also, $J_n = 3\alpha_1^n$ and $\alpha_1^3 = t$.

If θ is a primitive m^{th} root of unity, the next lemma follows easily from (2.2).

Lemma 2.2: For $j = 1, \ldots, k$, let $c_{m,jm}$ be defined by (2.2). Then

$$c_{m,km} = (-1)^{(k+1)m} e_k^m.$$

In particular, for the recurrence (1.1), we have $k = 3$, $e_3 = t$, and $c_{m,3m} = t^m$.

3. THE MAIN RESULT

We first prove two lemmas which are of interest in their own rights.

Lemma 3.1: Let $\{V_n\}$ and $\{W_n\}$ be defined by (1.1) and (1.4), respectively. Then, for all integers m and n such that $m \geq 1$,

$$V_{n+2m} = c_{m,m}V_{n+m} - c_{m,2m}V_n + t^m V_{n-m},$$

where the numbers $c_{m,m}$ and $c_{m,2m}$ are such that

$$W_m c_{m,m} = W_{2m} - t^m W_{-m},$$
$$t^{-m} W_{-m} c_{m,2m} = W_{-2m} - t^{-m} W_m.$$

Proof: In (2.1), let $a_j = W_j$, $k = 3$, $e_1 = r$, $e_2 = s$, $e_3 = t$. By (2.3) and Lemma 2.2, we have

$$W_{n+2m-h} = c_{m,m}W_{n+m-h} - c_{m,2m}W_{n-h} + t^m W_{n-m-h}. \tag{3.1}$$

Putting $h = n$ into (3.1), we have

$$W_{2m} = c_{m,m}W_m + t^m W_{-m},$$

and putting $h = n + m$ into (3.1), we have

$$W_m = -c_{m,2m}W_{-m} + t^m W_{-2m}.$$

This completes the proof. □

Lemma 3.2: Let $\{V_n\}$, $\{J_n\}$, and $\{W_n\}$ be defined by (1.2), (1.3), and (1.4), respectively. If m and n are integers such that $m \geq 1$, then

$$V_{n+2m} = c_{m,m}V_{n+m} - c_{m,2m}V_n + t^m V_{n-m},$$

where

$$c_{m,m} = \frac{W_{2m} - t^m W_{-m}}{W_m} = J_m,$$

if $W_m \neq 0$, and

$$t^{-m} c_{m,2m} = \frac{W_{-2m} - t^{-m} W_m}{W_{-m}} = J_{-m},$$

if $W_{-m} \neq 0$.

Proof: It is clear from Lemma 3.1 that if $W_m \neq 0$, then

$$c_{m,m} = \frac{W_{2m} - t^m W_{-m}}{W_m}.$$

We will prove the lemma by factoring out W_m from $W_{2m} - t^m W_{-m}$, and getting $W_{2m} - t^m W_{-m} = W_m J_m$. Let α_1, α_2, α_3 be the roots of $x^3 - rx^2 - sx - t = 0$, where r, s, t are arbitrary integers, $t \neq 0$.

Case 1: $\alpha_1 \neq \alpha_2$; $\alpha_1 \neq \alpha_3$; $\alpha_2 \neq \alpha_3$. Then $\alpha_1\alpha_2\alpha_3 = t$ and, by Lemma 2.1,

$$W_n = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n,$$

with $A_1 + A_2 + A_3 = 0$. We have

$$\begin{aligned}
W_{2m} - t^m W_{-m} &= W_{2m} - t^m\left(A_1\alpha_1^{-m} + A_2\alpha_2^{-m} + A_3\alpha_3^{-m}\right) \\
&= W_{2m} - A_1(\alpha_2\alpha_3)^m - A_2(\alpha_1\alpha_3)^m - A_3(\alpha_1\alpha_2)^m \\
&= A_1\alpha_1^{2m} + A_2\alpha_2^{2m} + A_3\alpha_3^{2m} + (A_2 + A_3)(\alpha_2\alpha_3)^m \\
&\quad + (A_1 + A_3)(\alpha_1\alpha_3)^m + (A_1 + A_2)(\alpha_1\alpha_2)^m \\
&= (A_1\alpha_1^m + A_2\alpha_2^m + A_3\alpha_3^m)(\alpha_1^m + \alpha_2^m + \alpha_3^m) \\
&= W_m J_m.
\end{aligned}$$

Case 2: $\alpha_1 = \alpha_2 \neq \alpha_3$. Then $\alpha_1^2\alpha_3 = t$, and by Lemma 2.1,

$$W_n = (A_1 + nA_2)\alpha_1^n + A_3\alpha_3^n,$$

with $A_1 + A_3 = 0$. We have

$$\begin{aligned}
W_{2m} - t^m W_{-m} &= W_{2m} - t^m(A_1 - mA_2)\alpha_1^{-m} - t^m A_3\alpha_3^{-m} \\
&= W_{2m} - (\alpha_1\alpha_3)^m(A_1 - mA_2) - A_3\alpha_1^{2m} \\
&= [(A_1 + 2mA_2)\alpha_1^{2m} + A_3\alpha_3^{2m}] - (\alpha_1\alpha_3)^m(A_1 - mA_2) + A_1\alpha_1^{2m} \\
&= 2(A_1 + mA_2)\alpha_1^{2m} + A_3\alpha_3^{2m} + (\alpha_1\alpha_3)^m(A_1 + mA_2) - 2A_1(\alpha_1\alpha_3)^m \\
&= 2(A_1 + mA_2)\alpha_1^{2m} + A_3\alpha_3^{2m} + (\alpha_1\alpha_3)^m(A_1 + mA_2) + 2A_3(\alpha_1\alpha_3)^m \\
&= [(A_1 + mA_2)\alpha_1^m + A_3\alpha_3^m](2\alpha_1^m + \alpha_3^m) \\
&= W_m J_m.
\end{aligned}$$

Case 3: $\alpha_1 = \alpha_2 = \alpha_3$. Then $\alpha_1^3 = t$ and, by Lemma 2.1,

$$W_n = (A_1 + nA_2 + n^2A_3)\alpha_1^n$$

with $A_1 = 0$. We have

$$\begin{aligned}
W_{2m} - t^m W_{-m} &= W_{2m} - t^m(-mA_2 + m^2A_3)\alpha_1^{-m} \\
&= W_{2m} + mA_2\alpha_1^{2m} - m^2A_3\alpha_1^{2m} \\
&= (2mA_2 + 4m^2A_3)\alpha_1^{2m} + mA_2\alpha_1^{2m} - m^2A_3\alpha_1^{2m} \\
&= (mA_2 + m^2A_3)\alpha_1^m(3\alpha_1^m) \\
&= W_m J_m.
\end{aligned}$$

In all of the above cases, m could be positive or negative. Thus, the proof is complete. □

Theorem 3.1: Let $\{V_n\}$ and $\{J_n\}$ be defined by (1.2) and (1.3), respectively. Then, for all integers m and n,

$$V_{n+2m} = J_m V_{n+m} - t^m J_{-m} V_n + t^m V_{n-m}. \tag{3.2}$$

Proof: Note that (3.2) is valid for $m=0$ and all n. Let m be a fixed positive integer.

Case 1: Suppose there is a sequence $\{W_n\}$, defined by (1.4), such that $W_m \neq 0$, and there is a (possibly different) sequence $\{W_n\}$ such that $W_{-m} \neq 0$ for given r, s, t. Lemma 3.2 can be applied, and (3.2) is valid for $m \geq 0$ and all integers n. Replacing n by $n-m$ in (3.2) and rearranging terms, we have

$$V_{n-2m} = J_{-m}V_{n-m} - t^{-m}J_mV_n + t^{-m}V_{n+m}.$$

Thus, if Lemma (3.2) can be applied, then (3.2) is valid for all integers m and n.

Case 2: Suppose that, for *every* sequence $\{W_n\}$ defined by (1.4), we have $W_m = 0$ (and also $W_{-m} = 0$, as we see below). For example, if $r=s=0$, and $m=3j$ for some integer j, then it is clear that $W_m = 0$ regardless of the values of W_1 and W_2. Another example is $r=1$, $s=-1$, $t=1$. If $m=4j$ for some integer j, then $W_m = 0$ regardless of the values of W_1 and W_2.

In Case 2, we first show that the roots α_1, α_2, α_3, defined in Lemma 2.1, must be distinct. If $\alpha_1 = \alpha_2$, then, by Lemma 2.1, initial conditions $W_0 = 0$, $W_1 = \alpha_1$, $W_2 = 2\alpha_1^2$ give $W_m = m\alpha_1^m \neq 0$. Thus, the roots are distinct and nonzero (since $\alpha_1\alpha_2\alpha_3 = t \neq 0$) and

$$W_n = C_1\alpha_1^n + C_2\alpha_2^n + C_3\alpha_3^n,$$

with $C_1 + C_2 + C_3 = 0$. Let $j=2$ or $j=3$. By Lemma 2.1, initial conditions $W_0 = 0$, $W_1 = \alpha_1 - \alpha_j$, and $W_2 = \alpha_1^2 - \alpha_j^2$ give $W_m = \alpha_1^m - \alpha_j^m$. Thus, we have

$$\alpha_1^m = \alpha_2^m = \alpha_3^m \quad \text{and} \quad \alpha_1^{-m} = \alpha_2^{-m} = \alpha_3^{-m}.$$

Note that this gives

$$t^m = (\alpha_1\alpha_2\alpha_3)^m = \alpha_1^{3m}. \tag{3.3}$$

Thus, in Case 2, if $\{V_n\}$ is defined by (1.2) and $\{W_n\}$ is defined by (1.4), for all integers n and m, we have

$$V_{n+m} = A_1\alpha_1^{n+m} + A_2\alpha_2^{n+m} + A_3\alpha_3^{n+m} = \alpha_1^m V_n. \tag{3.4}$$

By Lemma 2.1, we also have

$$J_m = 3\alpha_1^m. \tag{3.5}$$

Thus, by (3.3), (3.4), and (3.5),

$$V_{n+2m} = \alpha_1^{2m}V_n = t^m\alpha_1^{-m}V_n = t^mV_{n-m},$$
$$J_mV_{n+m} = 3\alpha_1^{2m}V_n = t^mJ_{-m}V_n,$$

and it is easy to see that (3.2) holds. The proof is now complete. □

As an example of Theorem 3.1, consider the Tribonacci sequence $\{P_n\}$ defined by

$$P_n = P_{n-1} + P_{n-2} + P_{n-3},$$

with P_0, P_1, and P_2 arbitrary complex numbers. For all n and m, we have

$$P_{n+2m} = S_mP_{n+m} - S_{-m}P_n + P_{n-m}, \tag{3.6}$$

where $S_0 = 3$, $S_1 = 1$, $S_2 = 3$ and, for all n, $S_n = S_{n-1} + S_{n-2} + S_{n-3}$. Thus,

$$
\begin{aligned}
P_{n+4} &= 3P_{n+2} + P_n + P_{n-2}, \\
P_{n+6} &= 7P_{n+3} - 5P_n + P_{n-3}, \\
P_{n+8} &= 11P_{n+4} + 5P_n + P_{n-4}, \\
P_{n+10} &= 21P_{n+5} + P_n + P_{n-5}.
\end{aligned}
$$

Formula (3.6) was a conjecture in [2].

FINAL COMMENTS

Once Theorem 3.1 is known, it can be proved by substituting the Binet forms (that is, the forms in Lemma 2.1) into formula (1.5). In this paper, however, we have tried to show how Theorem 3.1 can be derived without prior knowledge of it.

Much of the notation in the present paper was suggested in [4]. Reference [4] also contains new results for the generalized Tribonacci sequence, as well as references to related work.

Formula (1.6) for the Fibonacci numbers has been generalized by Horadam [1] in the following way. Let a, b, s, and t be integers, with $t \neq 0$, and define the sequence $\{v_n\} = \{v_n(a, b; s, t)\}$ by $v_0 = a$, $v_1 = b$, and $v_n = sv_{n-1} - tv_{n-2}$. Let $j_n = v_n(2, s; s, t)$. Then, for all integers n and m,

$$v_{n+m} = j_m v_n - t^m v_{n-m}, \tag{3.7}$$

which can be compared to (1.6). Using basically Theorem 2.1 of the present paper, the writer gave another proof of (3.7) in [2].

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