THE POWER OF 2 DIVIDING THE COEFFICIENTS OF CERTAIN POWER SERIES

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(Submitted August 1999-Final Revision January 2000)

1. INTRODUCTION

In this paper we prove a conjecture of Barrucand [1] concerning the highest power of 2 dividing the coefficients of certain formal power series. We also prove a more general result, and we give examples involving the convolved Fibonacci numbers, the convolved generalized Fibonacci numbers, and the Bernoulli numbers (of the first and second kinds) of higher order. The writer believes that all of the results are new.

With the definitions of $\upsilon(r)$ and $S(n)$ given below (Definitions 2.1 and 2.2, respectively), the conjecture can be stated as follows.

Theorem 1.1 (Barrucand's Conjecture): Let

$$H(x)=\sum_{n=0}^{\infty}h_n x^n$$

be a formal power series with $h_0=1$, $\upsilon(h_1)=1$, and $\upsilon(h_n)\geq 1$ for $n\geq 1$. Define a_n by

$$[H(2x)]^{1/2}=\sum_{n=0}^{\infty}a_n x^n.$$

Then $a_0=1$, and $\upsilon(a_n)=S(n)$ for $n>0$.

Our main result, Theorem 4.1, generalizes Theorem 1.1 to series of the form $[H(2^k x)]^r$, where r is rational and 2^k $(k\geq 1)$ is the highest power of 2 dividing the denominator of r.

A summary by sections follows. Section 2 is a preliminary section in which we state the basic definitions and lemmas that are needed. In Section 3 we furnish a proof of Theorem 1.1, and give three examples. The first example provides a new proof of a well-known formula for the highest power of 2 dividing $n!$. The second example, involving a theta function, was the original motivation for Theorem 1.1. The third example determines a formula for the highest power of 2 dividing the n^{th} Catalan number. In Section 4 we prove the main result, Theorem 4.1, which generalizes Theorem 1.1. In Section 5 we give examples of Theorem 4.1 that involve Fibonacci numbers and Bernoulli numbers.

2. PRELIMINARIES

We use the following definitions of $\upsilon(r)$ and $S(n)$.

Definition 2.1: Let $r=u/v$ be a rational number with $\gcd(u,v)=1$. Define $\upsilon(r)$ to be the exponent of the highest power of 2 that divides r. That is, if $\upsilon(r)>0$, then $2^{\upsilon(r)}\,\|\,u$; if $\upsilon(r)<0$, then $2^{-\upsilon(r)}\,\|\,v$; if $\upsilon(r)=0$, then u and v are both odd.

Definition 2.2: Let n be a positive integer with the following base 2 representation:

$$n = n_0 + n_1 2 + n_2 2^2 + \cdots + n_k 2^k \quad \text{(each } n_i = 0 \text{ or } 1\text{)}.$$

Define $S(n) = n_0 + n_1 + \cdots + n_k$, the sum of the binary digits of n.

All infinite series in this paper are "formal" with rational coefficients. A good reference for the theory of formal power series, and the properties of such series, is [5]. Of particular relevance for this paper is the binomial theorem (Theorem 17 in [5]), which can be stated as follows.

Lemma 2.1: Let r be a rational number, and let $F(x)$ be the formal power series:

$$F(x) = 1 + \sum_{n=1}^{\infty} f_n x^n = 1 + F_1(x).$$

Then

$$[F(x)]^r = 1 + \sum_{n=1}^{\infty} \binom{r}{n} [F_1(x)]^n,$$

where $\binom{r}{n} = r(r-1)\cdots(r-n+1)/n!$.

We need the next lemma for Example 3.1 in the next section.

Lemma 2.2: Let n be a positive integer, and $\upsilon(n)$ be defined by Definition 2.1. Then $\upsilon(n!) < n$.

Proof: We use induction on n. Clearly Lemma 2.2 is true for $n = 1$ and $n = 2$; assume that $\upsilon(j!) < j$ for $j = 1, 2, \ldots, n-1$. If n is even, let $n = 2m$ $(m > 1)$, so

$$n! = [1 \cdot 3 \cdots (2m-1)][2 \cdot 4 \cdots 2m] = [1 \cdot 3 \cdots (2m-1)]2^m m!. \tag{2.1}$$

By the induction hypothesis, $\upsilon(m!) < m$, so by (2.1) we have $\upsilon(n!) < 2m$; that is, $\upsilon(n!) < n$. The proof for odd n is entirely similar. □

3. PROOF OF BARRUCAND'S CONJECTURE

We use the notation in the statement of Theorem 1.1. It is clear from Lemma 2.1 that $a_0 = 1$; in fact, we could use the binomial theorem to determine a_1 and a_2 as well, but the following combinatorial approach is more useful. Since

$$H(2x) = \left(\sum_{n=0}^{\infty} a_n x^n \right)^2,$$

we have

$$2^n h_n = \sum_{k=0}^{n} a_k a_{n-k},$$

so that

$$a_n = 2^{n-1} h_n - \frac{1}{2} \sum_{j=1}^{n-1} a_j a_{n-j} \quad (n \geq 1). \tag{3.1}$$

Thus, $a_1 = h_1$ and $a_2 = 2h_2 - \frac{1}{2}(a_1)^2$, and the conjecture is true for $n = 0, 1, 2$. We now use induction on n in equation (3.1). Assume $\upsilon(a_j) = S(j)$ for $j = 1, \ldots, n-1$, and define M as follows:

$$M = \text{the minimum value of } \upsilon(a_j a_{n-j}) \text{ for } j = 1, \ldots, n-1.$$

Case 1: $n = 2^t$, with $t > 1$. It is clear that $M = 2 = \upsilon(a_j a_j)$, where $j = 2^{t-1}$. Thus, by (3.1), $\upsilon(a_n) = 1 = S(n)$.

Case 2: $n = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_t}$, with $0 \le e_1 < e_2 < \cdots < e_t$, and $n \ne 2^{e_t}$. It is clear from the induction hypothesis that M is at least equal to t. In fact, it is clear (because of no "carries") that M occurs when

$$j = c_1 2^{e_1} + c_2 2^{e_2} + \cdots + c_t 2^{e_t} \quad (\text{each } c_i = 0 \text{ or } 1).$$

There are $2^t - 2$ such terms (the -2 represents the cases $c_i = 0$ for all i and $c_i = 1$ for all i). Thus,

$$M = S(j) + S(n-j) = S(n),$$

and by (3.1),

$$2^{-M} a_n \equiv -\frac{1}{2}(2^t - 2) \equiv 1 \pmod 2;$$

that is, $\upsilon(a_n) = S(n)$. This completes the proof. □

The following examples provide some motivation for Theorem 1.1. The first example is a new proof of a well-known and useful formula for $\upsilon(n!)$.

Example 3.1: Let

$$H(x) = e^{2x} = 1 + 2x + \sum_{n=2}^{\infty} \frac{2^n}{n!} x^n.$$

By Lemma 2.2, we know that $\upsilon(2^n/n!) \ge 1$ for $n \ge 1$, so we can apply Theorem 1.1 to $H(x)$. Since $[H(2x)]^{1/2} = e^{2x}$, Theorem 1.1 says that $\upsilon(2^n/n!) = S(n)$ for $n \ge 1$; that is, $\upsilon(n!) = n - S(n)$.

The second example was the original motivation for Theorem 1.1. It involves the theta function (see [7], Chapter 20):

$$\theta(x) = 1 + 2\sum_{n=1}^{\infty} x^{n^2}. \tag{3.2}$$

Example 3.2: Let $\theta(x)$ be defined by (3.2). Theorem 1.1 says that

$$[\theta(2x)]^{1/2} = 1 + \sum_{n=1}^{\infty} 2^{S(n)} I_n x^n,$$

where I_n is rational and $\upsilon(I_n) = 0$. In fact, by Lemma 2.1, I_n is an odd *integer*.

Barrucand [1] has pointed out that $\theta(x)$ is a modular form, and it is very striking that the determination of the 2-valuation for $\sqrt{\theta(2x)}$ is so simple.

Example 3.3: Let $C_n = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number, with generating function (see [8], p. 82):

$$\frac{1 - \sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n.$$

If we let $H(x) = 1 - 2x$, then

$$[H(2x)]^{1/2} = 1 - \sum_{n=0}^{\infty} 2C_n x^{n+1},$$

and Theorem 1.1 says that $2^{S(n+1)-1}$ is the highest power of 2 dividing C_n.

4. GENERALIZATION OF BARRUCAND'S CONJECTURE

Throughout this section we assume $H(x)$ is defined by

$$H(x)=\sum_{n=0}^{\infty}h_nx^n,$$

with $h_0=1$, $\upsilon(h_1)=1$, and $\upsilon(h_n)\geq 1$ for $n\geq 1$.

The proof of the main result, Theorem 4.1, depends on the following two lemmas.

Lemma 4.1: For $k\geq 1$, define $h_{n,k}$ by

$$[H(2^kx)]^{2^{-k}}=\sum_{n=0}^{\infty}h_{n,k}x^n.$$

Then $h_{0,k}=1$, and $\upsilon(h_{n,k})=S(n)$ for $n\geq 1$.

Proof: The proof is by induction on k. By Theorem 1.1, we know the lemma is true for $k=1$. Assume it is true for a fixed $k\geq 1$, so that $h_{0,k}=1$ and $\upsilon(h_{n,k})=S(n)$ for $n\geq 1$. Therefore, $\upsilon(h_{1,k})=1$, and $\upsilon(h_{n,k})\geq 1$ for $n\geq 1$. Now we can use Theorem 1.1 again, starting with

$$[H(2^kx)]^{2^{-k}}=\sum_{n=0}^{\infty}h_{n,k}x^n$$

instead of $H(x)$, to get $h_{0,k+1}=1$ and $\upsilon(h_{n,k+1})=S(n)$. This completes the proof. $\square$

Lemma 4.2: Let $r=\frac{u}{w}$, where u and w are odd integers with $\gcd(u,w)=1$. Define $h_n^{(r)}$ by

$$[H(x)]^r=\sum_{n=0}^{\infty}h_n^{(r)}x^n. \tag{4.1}$$

Then $h_0^{(r)}=1$, $\upsilon(h_1^{(r)})=1$, and $\upsilon(h_n^{(r)})\geq 1$ for $n\geq 1$.

Proof: We first consider the case when $w=1$. Since u is an odd integer (positive or negative), it is clear from Lemma 2.1 that $h_0^{(u)}=1$, $\upsilon(h_1^{(u)})=\upsilon(uh_1)=1$, and $\upsilon(h_n^{(u)})\geq 1$ for $n\geq 1$.

Barrucand [1] has pointed out that it is possible to use Lemma 2.1 to get the desired result for $r=\frac{u}{w}$ $(w\geq 1)$. However, we present here a simple combinatorial proof. Suppose $w>1$, and let $H_1(x)=[H(x)]^u$. For convenience, and to make the rest of the proof more readable, we will use the notation $g_n=h_n^{(u/w)}$. Thus, we can write

$$[H_1(x)]^{1/w}=[H(x)]^{u/w}=\sum_{n=0}^{\infty}g_nx^n$$

and

$$H_1(x)=\left(\sum_{n=0}^{\infty}g_nx^n\right)^w. \tag{4.2}$$

We see that $1=(g_0)^w$, which implies $g_0=1$. By (4.2), we also have $h_1^{(u)}=wg_1$, which implies $\upsilon(g_1)=\upsilon(h_1^{(u)})=1$. For $n\geq 1$, we have, by (4.2),

$$h_n^{(u)}=\sum g_{n_1}g_{n_2}\cdots g_{n_w}, \tag{4.3}$$

where the summation is over all $n_1,n_2,\ldots,n_w$ such that $0\leq n_i\leq n$ $(i=1,\ldots,w)$, and $n_1+n_2+\cdots+n_w=n$. We rewrite (4.3) in the form

$$wg_n = h_n^{(u)} - \sum g_{n_1} g_{n_2} \cdots g_{n_w},$$

with $0 \le n_i < n$ $(i = 1, \ldots, w)$. It is now clear that we can use induction on n to prove that $2 \mid g_n$, for $n \ge 1$. Thus, $\upsilon(g_n) \ge 1$ for $n \ge 1$. This completes the proof. □

Theorem 4.1: Let

$$H(x) = \sum_{n=0}^{\infty} h_n x^n$$

with $h_0 = 1$, $\upsilon(h_1) = 1$, and $\upsilon(h_n) \ge 1$ for $n \ge 1$. Let $r = u/v$ be a rational number with gcd $(u, v) = 1$, and suppose $2^k \| v$, with $k \ge 1$. Define $h_{n,k}^{(r)}$ by

$$[H(2^k x)]^r = \sum_{n=0}^{\infty} h_{n,k}^{(r)} x^n.$$

Then $h_{0,k}^{(r)} = 1$, and $\upsilon(h_{n,k}^{(r)}) = S(n)$ for $n \ge 1$.

Proof: Let $v = 2^k w$, so w is an odd integer. By Lemma 4.2, we know that the numbers g_n defined by

$$[H(x)]^{u/w} = \sum_{n=0}^{\infty} g_n x^n$$

have the properties $g_0 = 1$, $\upsilon(g_1) = 1$, and $\upsilon(g_n) \ge 1$ for $n \ge 1$. Thus, if we use $[H(x)]^{u/w}$ instead of $H(x)$ in Lemma 4.1, the proof of Theorem 4.1 follows immediately. □

5. EXAMPLES

In all of the examples, we assume r is defined as follows:

$$r = \frac{u}{v} \text{ is a rational number (positive or negative); } \gcd(u, v) = 1;\ 2^k \| v \text{ for } k \ge 1. \tag{5.1}$$

Example 5.1 (Convolved Fibonacci Numbers): Let

$$H(x) = \frac{1}{1 - 2x - 4x^2} = \sum_{n=0}^{\infty} 2^n F_{n+1} x^n,$$

where F_{n+1} is the Fibonacci number. Note that $F_1 = 1$, $\upsilon(2F_2) = 1$, and $\upsilon(2^n F_{n+1}) \ge 1$ for $n \ge 1$. Thus, we can apply Theorem 4.1 to $H(x)$ to get results for the "convolved" Fibonacci numbers of rational order, $F_{n+1}^{(j)}$ (see [8], p. 89). These numbers are defined by

$$\left(\frac{1}{1 - x - x^2}\right)^j = \sum_{n=0}^{\infty} F_{n+1}^{(j)} x^n,$$

and Theorem 4.1 gives the following result for $n \ge 1$ and r defined by (5.1):

$$\upsilon(F_{n+1}^{(r)}) = S(n) - (k+1)n.$$

Example 5.2 (Generalized Convolved Fibonacci Numbers): Let p, q, and b be rational numbers such that $\upsilon(p) \ge 0$, $\upsilon(q) \ge 0$, and $\upsilon(b) = 0$. That is, p and q are 2-integral, and b is the quotient of two odd integers. Consider the following generalization of the Fibonacci numbers. Let $w_1 = 1$, $w_2 = b$, and, for $n \ge 3$,

$$w_n = pw_{n-1} - qw_{n-2}. \tag{5.2}$$

We define the generalized convolved Fibonacci numbers, $w_{n+1}^{(j)}$, by means of

$$\left(\frac{1+(b-p)x}{1-px+qx^2}\right)^j = \sum_{n=0}^{\infty} w_{n+1}^{(j)} x^n.$$

To the writer's knowledge, these numbers have not been studied before. Note that $w_{n+1}^{(1)} = w_{n+1}$; also, if $p=1$, $q=-1$, and $b=1$, then $w_{n+1} = F_{n+1}$. More generally, we see that $w_1 = 1$, $\upsilon(2w_2) = 1$, and, by recurrence (5.2), it is clear that $\upsilon(2^n w_{n+1}) \geq 1$ for $n \geq 1$. Thus, we can start with

$$H(x) = \frac{1+(b-p)(2x)}{1-2px+4qx^2} = \sum_{n=0}^{\infty} 2^n w_{n+1} x^n,$$

and apply Theorem 4.1 to get the following result for $n \geq 1$ and r defined by (5.1):

$$\upsilon(w_{n+1}^{(r)}) = S(n) - (k+1)n.$$

Example 5.3 (Bernoulli Numbers of Higher Order): The Bernoulli number of higher (rational) order, $B_n^{(j)}$, may be defined by means of

$$\left(\frac{x}{e^x-1}\right)^j = \sum_{n=0}^{\infty} B_n^{(j)} \frac{x^n}{n!}.$$

These numbers were evidently first introduced by Nörlund (see [6], Ch. 6), and they have been the subject of many papers (for integer j especially). The numbers $B_n = B_n^{(1)}$ are the ordinary Bernoulli numbers, and the following facts are well known: $B_0 = 1$, $B_1 = -1/2$, $B_n = 0$ if n is odd ($n > 1$), and $\upsilon(B_{2n}) = -1$ if $n \geq 1$. Thus, we can start with

$$H(x) = \frac{4x}{e^{4x}-1} = \sum_{n=0}^{\infty} 4^n B_n \frac{x^n}{n!},$$

and apply Theorem 4.1 to get the following result for $n \geq 1$ and r defined by (5.1):

$$\upsilon(B_n^{(r)}) = -(k+1)n.$$

We note that this is a special case of a theorem of Carlitz [2] for the Nörlund polynomial $B_n^{(x)}$.

Example 5.4 (Higher-Order Bernoulli Numbers of the Second Kind): The higher-order Bernoulli number of the second kind, $b_n^{(j)}$, can be defined by

$$\left(\frac{x}{\log(1+x)}\right)^j = \sum_{n=0}^{\infty} b_n^{(j)} x^n.$$

The numbers $b_n^{(1)} = b_n$ are the Bernoulli numbers of the second kind defined by Jordan [4]. It is known [3] that $b_0 = 1$, $b_1 = 1/2$, and $\upsilon(b_n) = -n$ for $n \geq 1$. To the author's knowledge, the numbers $b_n^{(j)}$ for $j \neq 1$ have not been studied. If we start with

$$H(x) = \frac{4x}{\log(1+4x)} = \sum_{n=0}^{\infty} 4^n b_n x^n,$$

and apply Theorem 4.1, we get the following result for $n \geq 1$ and r defined by (5.1):

$$\upsilon(b_n^{(r)}) = S(n) - (k+2)n.$$

6. FINAL COMMENTS

The writer thanks P. Barrucand for his many insights and suggestions.

The simple combinatorial methods of this paper do not seem to work for primes other than 2. Thus, the problem of determining the highest power of 3 (or any other odd prime) dividing numbers generated by a rational power of a generating function is, in general, difficult, and it will probably require a different approach.

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AMS Classification Numbers: 05A15, 11B39

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