SYMBIOTIC NUMBERS ASSOCIATED WITH IRRATIONAL NUMBERS

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INTRODUCTION

The *upper symbiotic number* $\mathcal{U}(R_1, R_2)$ of two linearly-ordered sets R_1 and R_2 is here introduced as the greatest number of elements of R_1 that lie strictly between consecutive elements of R_2. Similarly, the *lower symbiotic number* $\mathfrak{L}(R_1, R_2)$ is the least number of elements of R_1 that lie strictly between consecutive elements of R_2. Henceforth in this paper, we shall consider only upper symbiotic numbers. We are interested in pairs of positive irrational numbers α and γ for which $\mathcal{U}(R_1, R_2)$ is finite, where $R_2 = \{i + j\alpha : i, j \in Z^+\}$, $R_1 = R_2 + \gamma$, and Z^+ denotes the nonnegative integers.

Let $s_n = i_n + j_n\alpha$ be the sequence obtained by arranging the elements of R_2 in increasing order. The main objective of this study can now be indicated specifically by this question: If γ is rationally independent of α, what is the greatest number of numbers of the form $i + j\alpha + \gamma$ that lie between consecutive numbers $i_n + j_n\alpha$ and $i_{n+1} + j_{n+1}\alpha$? The special case $\alpha = (1+\sqrt{5})/2$, along with some possibly new appearances of Fibonacci numbers, are considered in Example 1 and just after Theorem 3.

1. CONVERGENTS AND THE SEQUENCE *s*

First, we recall the notation of continued fractions: Write $\alpha = [\![a_0, a_1, a_2, \ldots]\!]$,

$$p_{-2} = 0,\ p_{-1} = 1,\ p_i = a_i p_{i-1} + p_{i-2},$$

and

$$q_{-2} = 1,\ q_{-1} = 0,\ q_i = a_i q_{i-1} + q_{i-2},$$

for $i \ge 0$. The numbers a_i, for $i \ge 0$, are the *partial quotients of* α, and the rational numbers p_i / q_i, for $i \ge -2$, are the *principal convergents of* α. For all nonnegative integers *i* and *j*, define

$$p_{i,j} = jp_{i+1} + p_i \quad \text{and} \quad q_{i,j} = jq_{i+1} + q_i. \tag{1}$$

The fractions $p_{i,j} / q_{i,j}$, for $1 \le j \le a_{i+2} - 1$, are the i^{th} *intermediate convergents of* α, and for $1 \le j \le a_{i+2} - 2$,

$$\cdots < \frac{p_i}{q_i} < \cdots < \frac{p_{i,j}}{q_{i,j}} < \frac{p_{i,j+1}}{q_{i,j+1}} < \cdots < \frac{p_{i+2}}{q_{i+2}} < \cdots < \alpha \quad \text{if } i \text{ is even} \tag{2}$$

and

$$\cdots > \frac{p_i}{q_i} > \cdots > \frac{p_{i,j}}{q_{i,j}} > \frac{p_{i,j+1}}{q_{i,j+1}} > \cdots > \frac{p_{i+2}}{q_{i+2}} > \cdots > \alpha \quad \text{if } i \text{ is odd.} \tag{3}$$

Note that, if $j = a_{i+2}$ in (1), then $p_{i,j} = p_{i+2}$ and $q_{i,j} = q_{i+2}$. This extension of the range of *j* will enable certain proofs to cover simultaneously the two cases, $1 \le j \le a_{i+2} - 1$ and $j = a_{i+2}$.

Let *s* denote the sequence whose terms, $s_n = i_n + j_n\alpha$, for $n = 0, 1, 2, \ldots$, are as given in the Introduction. A difference $|p - q\alpha|$ *first occurs at* s_n if

$$s_n - s_{n-1} = |p - q\alpha| \quad \text{and} \quad s_m - s_{m-1} \neq |p - q\alpha| \tag{4}$$

for all $m < n$, and *last occurs at* s_n if conditions (4) hold for all $m > n$. (In these definitions, p/q need not be a convergent to α.)

Define $\Delta_1 = s_1 - s_0$. Let n_2 be the least n such that $s_n - s_{n-1} \neq \Delta_1$, and let $\Delta_2 = s_{n_2} - s_{n_2-1}$. Continue inductively, so that n_h is, for each $h \geq 2$, the least n such that $s_n - s_{n-1}$ is not among the numbers $\Delta_1, \Delta_2, \ldots, \Delta_{h-1}$, and $\Delta_h = s_{n_h} - s_{n_h-1}$.

It will be helpful to provide single indexing for the doubly-indexed numbers p_{ij} and q_{ij}, as follows. Let $P_j = p_{0,j}$ and $Q_j = q_{0,j}$ for $j = 0, 1, \ldots, a_2 - 1$, and for $w = 1, 2, \ldots$, let

$$P_{a_2+a_3+\cdots+a_{w+1}+j} = p_{w,j} \quad \text{and} \quad Q_{a_2+a_3+\cdots+a_{w+1}+j} = q_{w,j}$$

for $j = 0, 1, \ldots, a_{w+2} - 1$. Below, $\lfloor x \rfloor$ represents the greatest integer $\leq x$, and the *fractional part of* x is given by $((x)) = x - \lfloor x \rfloor$.

Lemma 1: Suppose i is even, $0 \leq j \leq a_{i+2} - 1$, and p and q are nonnegative integers such that $0 < -p + q\alpha < -p_{ij} + q_{ij}\alpha$; then $q \geq q_{i,j+1}$ if $j < a_{i+2} - 1$, and $q \geq q_{i+2}$ if $j = a_{i+2} - 1$. Suppose i is odd, $0 \leq j \leq a_{i+2} - 1$, and p and q are nonnegative integers such that $0 < p - q\alpha < p_{ij} - q_{ij}\alpha$; then $p \geq p_{i,j+1}$ if $j < a_{i+2} - 1$, and $p \geq p_{i+2}$ if $j = a_{i+2} - 1$.

Proof: In case i is even, p_{ij}/q_{ij} is a best lower approximate to α, as proved in [1], so that $q > q_{ij}$. There are two cases:

Case 1: $j < a_{i+2} - 1$. Here, $p_{i,j+1}/q_{i,j+1}$ is a best lower approximate to α; since $q > q_{ij}$, we have $q \geq q_{i,j+1}$.

Case 2: $j = a_{i+2} - 1$. Here, p_{i+2}/q_{i+2} is a best lower approximate to α; since $q > q_{ij}$, we have $q \geq q_{i+2}$.

If i is odd, then p_{ij}/q_{ij} is a best upper approximate, and the asserted inequalities follow. □

Lemma 2: The differences Δ_h are given in three cases:

Case 1: $\alpha < 1$. Here, $\Delta_h = |P_{h-1} - Q_{h-1}\alpha|$ for $h = 1, 2, \ldots$.

Case 2: $\alpha > 1$ *and* $((\alpha)) > 1/2$. Here, $\Delta_1 = 1$ and $\Delta_h = |P_{h-2} - Q_{h-2}\alpha|$ for $h = 2, 3, \ldots$.

Case 3: $\alpha > 1$ *and* $((\alpha)) < 1/2$. In this case, $\Delta_1 = 1$, $\Delta_2 = ((\alpha))$,

$$\Delta_h = 1 - (((h-2)\alpha)) \quad \text{for } h = 3, 4, \ldots, a_1 + 1,$$

and Δ_h has the form $|P_k - Q_k\alpha|$ for all $h \geq a_1 + 2$.

Proof: First, suppose i is an even nonnegative integer. It is easy to check that (3) implies

$$\cdots > -p_{i0} + q_{i0}\alpha > -p_{i1} + q_{i1}\alpha > \cdots > -p_{i,a_{i+2}-1} + q_{i,a_{i+2}-1}\alpha \tag{5}$$

and (2) implies

$$\cdots > p_{i+1,0} - q_{i+1,0}\alpha > p_{i+1,1} - q_{i+1,1}\alpha > \cdots > p_{i+1,a_{i+3}-1} - q_{i+1,a_{i+3}-1}\alpha; \tag{6}$$

in the former case, for example, (3) implies $p_{i+1} > q_{i+1}\alpha$, so that

$$(j+1)p_{i+1} + p_i - (jp_{i+1} + p_i) > ((j+1)q_{i+1} + q_i - (jq_{i+1} + q_i))\alpha,$$

i.e., $p_{i,j+1} - p_{ij} > (q_{i,j+1} - q_{ij})\alpha$, so that $-p_{ij} + q_{ij}\alpha > -p_{i,j+1} + q_{i,j+1}\alpha$ as desired in (5); chain (6) likewise follows from (2).

Since $p_{i+2}/q_{i+2} < \alpha < p_{i+3}/q_{i+3}$, we also have

$$-p_{i,a_{i+2}-1} + q_{i,a_{i+2}-1}\alpha > p_{i+1,0} - q_{i+1,0}\alpha \tag{7}$$

and

$$p_{i+1,a_{i+3}-1} - q_{i+1,a_{i+3}-1}\alpha > -p_{i+2,0} + q_{i+2,0}\alpha. \tag{8}$$

Inequalities (5)-(8) are clearly equivalent to the chain

$$((\alpha)) = -p_0 + q_0\alpha = |P_0 - Q_0\alpha| > |P_1 - Q_1\alpha| > |P_2 - Q_2\alpha| > \cdots. \tag{9}$$

The numbers p_{ij}/q_{ij}, alias P_h/Q_h, comprise the complete set of best lower and upper approximates to α. Consequently, any difference Δ_h not included in chain (9) must exceed $((\alpha))$. We consider the following cases:

Case 1: $\alpha < 1$. Clearly $s_1 = \alpha$, so that $\Delta_1 = s_1 - s_0 = \alpha = ((\alpha))$. No difference Δ_h can exceed Δ_1 since, for any $s_n = u + v\alpha$ where $v \geq 1$, we have

$$s_n - s_{n-1} \leq s_n - (u + (v-1)\alpha) = ((\alpha)),$$

and, for $s_n = u + 0\alpha$, we have

$$s_n - s_{n-1} \leq s_n - ((u-1) + \lfloor 1/\alpha \rfloor \alpha) = 1 - \lfloor 1/\alpha \rfloor \alpha < \alpha.$$

Thus, $\Delta_h = |P_{h-1} - Q_{h-1}\alpha|$ for $h = 1, 2, \ldots$.

Case 2: $\alpha > 1$ *and* $((\alpha)) > 1/2$. Write $m = \lfloor \alpha \rfloor$. Then $s_i = i$ for $i = 1, 2, \ldots, m$, and $s_{m+1} = \alpha$. Consequently, $\Delta_1 = 1$ and $\Delta_2 = \alpha - m = ((\alpha))$. Write $m = s_m$ and $\alpha = s_{m+1}$, and, for any $n \geq m$, write $s_n = u + v\alpha$. If $u \geq m$, then

$$s_{n+1} - s_n \leq u - m + (v+1)\alpha - s_n = ((\alpha)),$$

whereas, if $u < m$, then

$$s_{n+1} - s_n \leq u + m + 1 + (v-1)\alpha - s_n = m + 1 - \alpha = 1 - ((\alpha)) < ((\alpha)).$$

Thus, $\Delta_h < \Delta_2$ for all $h \geq 3$, so that $\Delta_h = |P_{h-2} - Q_{h-2}\alpha|$ for $h = 2, 3, \ldots$.

Case 3: $\alpha > 1$ *and* $((\alpha)) < 1/2$. As in Case 2, clearly $\Delta_1 = 1$ and $\Delta_2 = \alpha - \lfloor \alpha \rfloor = ((\alpha))$. It is easy to check that $((j\alpha)) = j((\alpha))$ for $j = 1, 2, \ldots, a_1 = \lfloor 1/(\alpha - a_0) \rfloor$.

We seek conditions under which terms $j\alpha$ and $\lfloor 1 + j\alpha \rfloor$ are not consecutive in s: suppose $j\alpha < u + v\alpha < \lfloor 1 + j\alpha \rfloor$. Equivalently, $(j-v)\alpha < u < 1 + \lfloor j\alpha \rfloor - v\alpha$. Such an integer u exists if and only if $\lfloor (j-v)\alpha \rfloor < \lfloor 1 + \lfloor j\alpha \rfloor - v\alpha \rfloor$ or, equivalently, $\lfloor (j-v)\alpha \rfloor < \lfloor j\alpha \rfloor - \lfloor v\alpha \rfloor$, or yet again, $(((j-v)\alpha)) < ((j\alpha)) - ((v\alpha))$. Since $0 < j - v < j$, this last equality, hence the nonexistence of u, is equivalent to the condition $j \geq a_1 + 1$. That is to say, each $\lfloor 1 + j\alpha \rfloor - j\alpha$, which is equal to $1 - ((j\alpha))$, is one of the differences Δ_h for $j = 1, 2, \ldots, a_1$, and this fails to be the case for all $j \geq a_1 + 1$.

Every difference Δ_h is necessarily of the form $|p - q\alpha|$. By Lemma 1, if $|p - q\alpha| < ((\alpha))$, then $|p - q\alpha|$ is one of the differences Δ_h. We have already seen that in addition to these Δ_h are the a_1 numbers $1 - ((j\alpha))$, for $j = 0, 1, \ldots, a_1 - 1$. In order to see that there is no other difference Δ_h, we consider two possibilities.

Case 3.1: $p-q\alpha>0$. Here, $|p-q\alpha|=1-((q\alpha))$, which exceeds $((\alpha))$ and is a difference Δ_h for $q=1,2,\ldots,a_1$, and for no other values, as already proved.

Case 3.2: $-p+q\alpha>0$. Here, $|p-q\alpha|=((q\alpha))$. If $((q\alpha))>((\alpha))$, then for any n we have $s_n<s_n+(-\lfloor\alpha\rfloor+\alpha)=s_n+((\alpha))<s_n+((q\alpha))$, so that $((q\alpha))$ is not a difference Δ_h. □

Theorem 1: Suppose α is a positive irrational number. If i is even and $0\le j\le a_{i+2}-1$, then $|p_{ij}-q_{ij}\alpha|$ first occurs at $q_{ij}\alpha$ and last occurs at $p_{i+1}-1+(q_{ij}+q_{i+1}-1)\alpha$. If i is odd and $0\le j\le a_{i+2}-1$, then $|p_{ij}-q_{ij}\alpha|$ first occurs at p_{ij} and last occurs at $p_{ij}+p_{i+1}-1+(q_{i+1}-1)\alpha$.

Proof: Suppose i is even, and let h be the index for which $\Delta_h=|p_{ij}-q_{ij}\alpha|$. Since i is even, $\Delta_h=-p_{ij}+q_{ij}\alpha$. Let m be the index such that $s_m=q_{ij}\alpha$, and suppose that Δ_h first occurs at $s_w=u+v\alpha$, where $w<m$. Then $s_{w-1}=u+p_{ij}+(v-q_{ij})\alpha$. Since s_{w-1} is a term of sequence s, we have $v\ge q_{ij}$, but then $u+v\alpha\ge q_{ij}\alpha$, contrary to $s_w<s_m$. Therefore, Δ_h does not occur before s_m. Next, we show that

$$s_m-s_{m-1}=\Delta_h. \tag{10}$$

Since $s_m=q_{ij}\alpha$, it suffices to prove that $s_{m-1}=p_{ij}$. Now $p_{ij}<q_{ij}\alpha$, since i is even. So, suppose, contrary to (10), that $p_{ij}<s_x=y+z\alpha<q_{ij}\alpha$ for some nonnegative integers x, y, z. Then $q_{ij}\alpha-p_{ij}>(q_{ij}-z)\alpha-y>0$, but this is untenable because p_{ij}/q_{ij} is a best lower approxi-mate to α. Therefore, (10) holds.

Turning now to the last occurrence of Δ_h, let n be the index such that $s_n=p_{i+1}-1+(q_{ij}+q_{i+1}-1)\alpha$. Then

$$s_n-\Delta_h=p_{i+1}+p_{ij}-1+(q_{i+1}-1)\alpha,$$

and this is clearly a number in the sequence s. We must show that, for any difference $\Delta=|-p_{kl}+Q_{kl}\alpha|$ less than Δ_h, the number $s_n-\Delta$ is not in s. (By Lemma 2, the only differences Δ that need be considered are, in fact, of the form $|-p_{kl}+q_{kl}\alpha|$.)

Case 1: *Even k*. Here $\Delta=-p_{kl}+q_{kl}\alpha$. By Lemma 2, we have $q_{kl}\ge q_{i,j+1}$ (which is q_{i+1} if $j=a_{i+2}-1$). Then

$$s_n-\Delta=p_{i+1}+p_{kl}-1+(q_{ij}+q_{i+1}-1-1_{kl})\alpha,$$

and the coefficient of α is $\le q_{ij}+q_{i+1}-1-q_{i,j+1}$, but since this number is -1, the number $s_n-\Delta$ is not in s.

Case 2: Odd k. Here, $\Delta=p_{kl}-q_{kl}\alpha$. By Lemma 2, we have $p_{kl}\ge p_{i,j+1}$ (which is p_{i+1} if $j=a_{i+2}-1$), and

$$s_n-\Delta=p_{i+1}-p_{kl}-1+(q_{ij}+q_{i+1}-1+q_{kl})\alpha.$$

Since $p_{i+1}-p_{kl}-1<0$, the number $s_n-\Delta$ is not in s.

We now know that $s_n-s_{n-1}=\Delta_h$. That is, Δ_h occurs at s_n. To see that this location in s marks the *last* occurrence of Δ_h, suppose $m>n$. We must show that $s_m-s_{m-1}<s_n-s_{n-1}$. Write s_m as $u+v\alpha$; then one of the following cases holds: (A) $u\ge p_{i+1}$; (B) $v\ge q_{ij}+q_{i+1}$.

Case A: $u\ge p_{i+1}$. Here, the number $u-p_{i+1}+(v+q_{i+1})\alpha$ is a term $s_{m'}$ in the sequence s. Since $-p_{i+1}+q_{i+1}\alpha<0$, we have $s_{m'}=u-p_{i+1}+(v+q_{i+1})\alpha<u+v\alpha=s_m$, so that $s_{m'}\le s_{m-1}$, and

$$s_m-s_{m-1}\le s_m-s_{m'}=u+v\alpha-(u-p_{i+1}+(v+q_{i+1}\alpha))=p_{i+1}-q_{i+1}\alpha<s_n-s_{n-1}.$$

Case B: $v \geq q_{ij} + q_{i+1}$. Here, $q_{ij} + q_{i+1} = (j+1)q_{i+1} + q_i = q_{i,\,j+1}$. Therefore, the number $u + p_{i,\,j+1} + (v - q_{i,\,j+1})\alpha$ is a term $s_{m'}$. Since $p_{i,\,j+1} - q_{i,\,j+1}\alpha < 0$, we have $s_{m'} \leq s_{m-1}$, so that

$$\begin{aligned} s_m - s_{m-1} \leq s_m - s_{m'} &= u + v\alpha - (u + p_{i,\,j+1} + (v - q_{i,\,j+1})\alpha) \\ &= -p_{i,\,j+1} + q_{i,\,j+1}\alpha < s_n - s_{n-1}. \end{aligned}$$

This finishes a proof for even i. A proof for odd i is similar and thus omitted here. □

Corollary 1.1: Suppose α is a positive irrational number. Let $n = n(h)$ be the index such that Δ_h last occurs at s_n. If $\alpha < 1$ or $((\alpha)) > 1/2$, the sequence $n(h)$ is strictly increasing. If $\alpha > 1$ and $((\alpha)) < 1/2$, the sequence $n(a_1+2)$, $n(a_1+3)$, $n(a_1+4)$, ... is strictly increasing.

Proof:

Case 1: i even. Assume first that $\alpha < 1$. If $0 \leq j \leq a_{i+2} - 2$, then, by Theorem 1, the difference $\Delta_h = -p_{ij} + q_{ij}\alpha$ last occurs at $n_1 = p_{i+1} - 1 + (q_{ij} + q_{i+1} - 1)\alpha$. By Lemma 2, we obtain $\Delta_{h+1} = -p_{i,\,j+1} + q_{i,\,j+1}\alpha$, which last occurs at $n_2 = p_{i+1} - 1 + (q_{i,\,j+1} + q_{i+1} - 1)\alpha$. Since $q_{i,\,j+1} > q_{ij}$, we have $n_2 > n_1$.

If $j = a_{i+2} - 1$, then the difference $\Delta_h = -p_{ij} + q_{ij}\alpha$ last occurs at $n_1 = p_{i+1} - 1 + (q_{ij} + q_{i+1} - 1)\alpha$ and, by Lemma 1, Δ_{h+1}, namely $p_{i+1} - q_{i+1}\alpha$, last occurs at $p_{i+1} + p_{i+2} - 1 + (q_{i+2} - 1)\alpha$. We have $(q_{i,\,a_{i+2}-1} + q_{i+1})\alpha < p_{i+2} + q_{i+2}\alpha$, so that

$$p_{i+1} - 1 + (q_{i,\,a_{i+2}-1} + q_{i+1} - 1)\alpha < p_{i+1} + p_{i+2} - 1 + (q_{i+2} - 1)\alpha,$$

which is to say that the last occurrence of $-p_{ij} + q_{ij}\alpha$ precedes that of $p_{i+1,\,0} - q_{i+1,\,0}\alpha$.

Next, assume that $\alpha > 1$ and $((\alpha)) > 1/2$. The difference $\Delta_1 = 1$ occurs at $\lfloor \alpha \rfloor$ and is easily seen not to occur thereafter. Clearly, this last occurrence of Δ_1 precedes the last appearance of $\Delta_2 = ((\alpha))$. The proof given above for the case $\alpha < 1$ now applies to all Δ_h for $h \geq 2$.

Finally, assume that $\alpha > 1$ and $((\alpha)) < 1/2$. By Case 3 of Lemma 2 and the method used in Case 1 above, the sequence $n(a_1+2)$, $n(a_1+3)$, $n(a_1+4)$, ... is strictly increasing.

Case 2: i odd. A proof much like that for i even is omitted. □

Corollary 1.2: Suppose α is a positive irrational number and $i \geq 0$. If the difference $|p_{ij} - q_{ij}\alpha|$ last occurs at s_n, then the difference $|p_{i+1} - q_{i+1}\alpha|$ first occurs before s_n, and no difference less than $|p_{i+2} - q_{i+2}\alpha|$ first occurs before s_n.

Proof: For the first assertion it suffices, by Theorem 1, to observe that, for even $i \geq 2$,

$$p_{i+1} < p_{i+1} - 1 + (q_{ij} + q_{i+1} - 1)\alpha,$$

and for odd i,

$$q_{i+1}\alpha < p_{ij} + p_{i+1} - 1 + (q_{i+1} - 1)\alpha.$$

For the second assertion, first suppose i is even. The well-known identity $p_{i+1}q_i - p_iq_{i+1} = 1$ implies that $a_{i+3}p_iq_{i+1} - p_{i+1}q_i \geq -1$, so that

$$\begin{aligned} a_{i+2}p_{i+1}q_{i+1}(a_{i+3} - 1) + a_{i+3}p_iq_{i+1} - p_{i+1}q_i + p_{i+1} + q_{i+1} &> 0, \\ q_{i+1}(a_{i+3}(a_{i+2}p_{i+1} + p_i) + 1) &> p_{i+1}(a_{i+2}q_{i+1} + q_i - 1), \\ q_{i+1}(a_{i+3}p_{i+2} + 1) &> p_{i+1}(q_{i+2} - 1), \end{aligned}$$

from which follows

$$\frac{p_{i+3}-p_{i+1}+1}{q_{i+2}-1} > \frac{p_{i+1}}{q_{i+1}} > \alpha$$

so that

$$\begin{aligned} p_{i+3} &> p_{i+1}-1+((a_{i+2}-1)q_{i+1}+q_i+q_{i+1}-1)\alpha \\ &\geq p_{i+1}-1+(q_{ij}+q_{i+1}-1)\alpha, \end{aligned}$$

and, by Theorem 1, the difference $p_{i+3}-q_{i+3}\alpha$ first occurs after s_n.

It is clear from Theorem 1 that, if a difference Δ first occurs after the difference $-p_{i+2}+q_{i+2}\alpha$ first occurs, then either $\Delta = p_{i+3}-q_{i+3}\alpha$ or else Δ first occurs after $p_{i+3}-q_{i+3}\alpha$ first occurs. We have therefore finished with a proof for even i. A proof for odd i involving the inequality

$$\frac{p_{i+2}-1}{a_{i+3}q_{i+2}+1} < \alpha$$

is similar and omitted. □

2. THE SPAN OF α

For $n = 1, 2, \ldots$, let

$$f(n) = \frac{\max\limits_{k\geq n}\{s_k - s_{k-1}\}}{\min\limits_{k\geq n}\{s_k - s_{k-1}\}}, \tag{11}$$

and define the *span of* α as

$$s(\alpha) = \sup\{f(n) : n = 0, 1, 2, \ldots\}.$$

Theorem 2: Suppose $\alpha = [\![a_0, a_1, \ldots]\!]$. Then the span of α is finite if and only if the partial quotients a_i are bounded.

Proof: Let n be large enough that the differences $\Delta_h = s_m - s_{m-1}$ are of the form $|p_{ij}-q_{ij}\alpha|$ for all $m \geq n$; this is possible by Lemma 2. Then, in (11), the numerator ranges through differences $\Delta_h = |p_{ij}-q_{ij}\alpha|$, where $i \geq 0$ and $1 \leq j \leq a_{i+2}-1$. Now suppose $|p_{ij}-q_{ij}\alpha|$ last occurs at s_n and $|p_{i+1}-q_{i+1}\alpha|$ last occurs at $s_{n'}$. Then $n' > n$ by Corollary 1.1, and using Corollary 1.2 we find

$$\frac{\max\limits_{k\geq n}\{s_k - s_{k-1}\}}{\min\limits_{k\leq n}\{s_k - s_{k-1}\}} = \frac{|p_{ij}-q_{ij}\alpha|}{\min\limits_{k\leq n}\{s_k - s_{k-1}\}} \leq \frac{|p_i - q_i\alpha|}{\min\limits_{k\leq n}\{s_k - s_{k-1}\}} \leq \frac{|p_i - q_i\alpha|}{\min\limits_{k\leq n'}\{s_k - s_{k-1}\}} \leq \frac{|p_i - q_i\alpha|}{|p_{i+2}-q_{i+2}\alpha|},$$

so that

$$s(\alpha) \leq \sup_{i\geq 0} \frac{|p_i - q_i\alpha|}{|p_{i+2}-q_{i+2}\alpha|}. \tag{12}$$

From the standard identity

$$|p_i - q_i\alpha| = \frac{1}{r_1 r_2 \ldots r_{i+1}}, \text{ where } r_k := [a_k, a_{k+1}, \ldots],$$

we have

$$\frac{|p_i - q_i\alpha|}{|p_{i+1}-q_{i+1}\alpha|} = a_{i+2} + 1/r_{i+3},$$

whence

$$\frac{|p_i - q_i\alpha|}{|p_{i+2} - q_{i+2}\alpha|} = (a_{i+2} + 1/r_{i+3})(a_{i+3} + 1/r_{i+4}),$$

so that

$$s(\alpha) \le \sup_{i\ge 0}(a_{i+2}+1)(a_{i+3}+1). \tag{13}$$

Thus, $s(\alpha)$ is finite if and only if the partial quotients a_j are bounded. □

Example 1: In case $\alpha = (1+\sqrt{5})/2 = [\![1, 1, 1, \ldots]\!]$, it is easy to verify the following results:

(a) All the convergents are principal convergents, and $p_i/q_i = F_{i+2}/F_{i+1}$, where F_k denotes the k^{th} Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$ for $k = 2, 3, \ldots$;

(b) $\Delta_h = |F_h - F_{h-1}\alpha| = \alpha^{-h-1}$ for $h = 1, 2, 3, \ldots$;

(c) Δ_h first occurs at s_n, where $n = (F_{h-1}+1)(F_h+1)/2$ for $h = 2, 3, 4, \ldots$, and

$$s_n = \begin{cases} F_{h-1}\alpha & \text{if } h \text{ is even and } \ge 2, \\ F_h & \text{if } h \text{ is odd;} \end{cases}$$

(d) Δ_h last occurs at s_n, where

$$n = \begin{cases} F_{h+4}(F_{h+1}-1)/2 & \text{if } h \text{ is even and } \ge 2, \\ 1 + F_{h+4}(F_{h+1}-1)/2 & \text{if } h \text{ is odd,} \end{cases}$$

and

$$s_n = \begin{cases} F_{h+1} - 1 + (F_{h+1}-1)\alpha & \text{if } h \text{ is even and } \ge 2, \\ F_{h+2} - 1 + (F_h - 1)\alpha & \text{if } h \text{ is odd;} \end{cases}$$

(e) $s(\alpha) = \alpha + 1 \doteq 2.618034$.

Using the upper bound $\sup_{i\ge 0}(a_{i+2} + 1/r_{i+3})(a_{i+3} + 1/r_{i+4})$ from the proof of Theorem 3, we have three more examples.

Example 2: $s(\sqrt{2}) = s([\![1, \overline{2}]\!]) \le \left(2 + \frac{1}{1+\sqrt{2}}\right)^2 = 31 - 12\sqrt{3} \doteq 10.2153.$

Example 3: $s(\sqrt{3}) = s([\![1, \overline{1, 2}]\!]) \le \left(1 + \frac{1}{\sqrt{3}+1}\right)\left(2 + \frac{2}{\sqrt{3}+1}\right) = 2 + \sqrt{3} \doteq 3.73205.$

Example 4: $s(\sqrt{5}) = s([\![2, \overline{4}]\!]) \le \left(4 + \frac{1}{\sqrt{5}-2}\right)^2 = 41 + 12\sqrt{5} \doteq 67.8328.$

3. UPPER SYMBIOTIC NUMBER FOR $\{i + j\alpha + \gamma\}$ AND $\{i + j\alpha\}$

We return now to the problem stated in the Introduction.

Theorem 3: Suppose α and γ are positive irrational numbers, γ rationally independent of α, and suppose α has bounded partial quotients. Let $R_2 = \{i + j\alpha : i, j \in Z^+\}$ and let $R_1 = R_2 + \gamma$. Then $U(R_1, R_2) \le s(\alpha) + 1$.

Proof: As before, let s_n denote the n^{th} largest number in R_2, after $s_0 = 0$, and assume now that at least one number t in R_1 lies between s_{n-1} and s_n. Let t_m and t_{m+w} be the least and greatest such numbers, where $w \geq 0$. We seek an upper bound for the number $w+1$ of numbers t between s_{n-1} and s_n. Let m' be the index for which $t_m = s_{m'} + \gamma$. The inequalities

$$s_{n-1} < t_m < t_{m+1} < \cdots < t_{m+w} < s_n$$

imply that

$$\begin{aligned} t_{m+w} - t_m &= (t_{m+1} - t_m) + (t_{m+2} - t_{m+1}) + \cdots + (t_{m+w} - t_{m+w-1}) \\ &= (s_{m'+1} - s_{m'}) + (s_{m'+2} - s_{m'+1}) + \cdots + (s_{m'+w} - s_{m'+w-1}) \\ &\geq w \min_{k \leq n} \{s_{m'+v} - s_{m'+v-1} : 1 \leq v \leq w\} \geq w \min_{k \leq n} \{s_k - s_{k-1}\}, \end{aligned}$$

so that

$$w \leq \frac{s_n - s_{n-1}}{\min_{k \leq n} \{s_k - s_{k-1}\}} \leq s(\alpha). \quad \square$$

Experimental sampling suggests that $\mathcal{U}(R_1, R_2) = 2$ for $\alpha = (1+\sqrt{5})/2$ regardless of the value of γ (as long as γ is positive, irrational, and rationally independent of α), and that similar results may hold for other quadratic irrationals. Sampling also suggests, perhaps unsurprisingly in view of the proof of Theorem 3, that $\mathcal{U}(R_1, R_2)$ may often be considerably less than the span of α.

We conclude with a particularly easy-to-state related unsolved problem. Let

$$R_2 = \{1, 3, 5, 9, 15\,25, 27, \ldots\} = \{3^i 5^j : i \geq 0, j \geq 0\}$$

and let

$$R_1 = \{2, 6, 10, 18, 30, 50, 54, \ldots\} = \{2r : r \in R_2\}.$$

Is $\mathcal{U}(R_1, R_2)$ finite? (For an equivalent formulation of this problem, let $R_2 = \{i + j\alpha\}$, where $\alpha = \log 5 / \log 3$, and let $R_1 = \gamma + R_2$, where $\gamma = \log 2 / \log 3$.)

REFERENCE

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