PENTAGONAL NUMBERS IN THE ASSOCIATED PELL SEQUENCE AND DIOPHANTINE EQUATIONS $x^2(3x-1)^2 = 8y^2 \pm 4$

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(Submitted May 1999-Final Revision February 2000)

1. INTRODUCTION

It is well known that a positive integer N is called a **pentagonal (generalized pentagonal) number** if $N = m(3m-1)/2$ for some integer $m > 0$ (for any integer m).

Luo Ming [2] has proved that 1 and 5 are the only pentagonal numbers in the Fibonacci sequence $\{F_n\}$, and later shown in [3] that 2, 1, and 7 are the only generalized pentagonal numbers in the Lucas sequence $\{L_n\}$.

In this paper we consider the **associated Pell sequence** $\{Q_n\}$ defined in [1] as

$$Q_0 = Q_1 = 1 \text{ and } Q_{n+2} = 2Q_{n+1} + Q_n \text{ for any integer } n \tag{1}$$

and establish that $Q_0 = Q_1 = 1$ and $Q_3 = 7$ are the only generalized pentagonal numbers in it.

2. PRELIMINARY RESULTS

We recall that the **Pell sequence** $\{P_n\}$ is defined by

$$P_0 = 0,\ P_1 = 1, \text{ and } P_{n+2} = 2P_{n+1} + P_n \text{ for any integer } n \tag{2}$$

and that it is closely related to the sequence $\{Q_n\}$. The following properties of these sequences are well known. For all integers n:

$$P_{-n} = (-1)^{n+1}P_n \text{ and } Q_{-n} = (-1)^n Q_n; \tag{3}$$

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \text{ and } Q_n = \frac{\alpha^n + \beta^n}{2}, \tag{4}$$

where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$;

$$Q_n^2 = 2P_n^2 + (-1)^n; \tag{5}$$

$$Q_{2n} = 2Q_n^2 - (-1)^n. \tag{6}$$

As a direct consequence of (4), we have

$$Q_{m+n} = 2Q_mQ_n - (-1)^n Q_{m-n} \text{ for all integers } m \text{ and } n. \tag{7}$$

The following congruence relation of $\{Q_n\}$ is very useful.

Lemma 1: If m is even and n, k are integers, then $Q_{n+2km} \equiv (-1)^k Q_n \pmod{Q_m}$.

Proof: If $k = 0$, the lemma is trivial. For $k > 0$, we use induction on k. By (7), $Q_{n+2m} = 2Q_{n+m}Q_m - (-1)^m Q_n$, which gives the lemma for $k = 1$ since m is even.

Assume that the lemma holds for all integers $\leq k$. Again by (7) and the induction hypothesis, we have

$$\begin{aligned} Q_{n+2(k+1)m} &= 2Q_{n+2km}Q_{2m} - Q_{n+2(k-1)m} \\ &\equiv 2(-1)^k Q_n Q_{2m} - (-1)^{k-1} Q_n \pmod{Q_m} \\ &\equiv (-1)^k (2Q_{2m}+1) Q_n \pmod{Q_m}. \end{aligned} \tag{8}$$

But since m is even, it follows from (6) that

$$2Q_{2m}+1 \equiv -1 \pmod{Q_m}. \tag{9}$$

Now (8) and (9) together prove the lemma for $k+1$. Hence, by induction, the lemma holds for $k>0$.

If $k<0$, say $k=-r$, where $r>0$, we have by (7) and (3) that

$$Q_{n+2km} = 2Q_nQ_{2rm} - Q_{n+2rm} \equiv 2Q_n(-1)^r - (-1)^r Q_n \pmod{Q_m} \equiv (-1)^r Q_n \pmod{Q_m}$$

which proves the lemma completely.

3. PENTAGONAL NUMBERS IN $\{Q_n\}$

Note that $N = m(3m-1)/2$ if and only if $24N+1 = (6m-1)^2$ so that N is generalized pentagonal if and only if $24N+1$ is the square of the form $6m-1$. Therefore, we have to first identify those n for which $24Q_n+1$ is a perfect square. We prove in this section that $24Q_n+1$ is a perfect square only when $n = 0, 1,$ or 3. We begin with

Lemma 2: Suppose $n \equiv 0$ or $1 \pmod{36}$. Then $24Q_n+1$ is a perfect square if and only if $n=0$ or 1.

Proof: If $n=0$ or 1, then $24Q_n+1=5^2$. Conversely, suppose $n\equiv 0$ or $1 \pmod{36}$. If $n \notin \{0,1\}$, then n can be written as $n = 2\cdot 3^2\cdot 2^r\cdot g+\varepsilon$, where $r\ge 1$, g is odd, and $\varepsilon = 0$ or 1. Write

$$m = \begin{cases} 3^2\cdot 2^r & \text{if } r\equiv 3 \text{ or } 8 \pmod{10}, \\ 3\cdot 2^r & \text{if } r \equiv 1 \text{ or } 6 \pmod{10}, \\ 2^r & \text{otherwise,} \end{cases}$$

so that $n = 2km+\varepsilon$, where k is odd.

Now, by Lemma 1 and (3), we have $24Q_n+1 = 24Q_{2km+\varepsilon}+1 \equiv 24(-1)^kQ_\varepsilon+1 \pmod{Q_m} \equiv 24(-1)+1 \pmod{Q_m} \equiv -23 \pmod{Q_m}$. Hence, the Jacobi symbol

$$\left(\frac{24Q_n+1}{Q_m}\right) = \left(\frac{-23}{Q_m}\right) = \left(\frac{-1}{Q_m}\right)\left(\frac{23}{Q_m}\right) = \left(\frac{-1}{Q_m}\right)\left(\frac{Q_m}{23}\right)\left(\frac{-1}{Q_m}\right) = \left(\frac{Q_m}{23}\right). \tag{10}$$

Also, since $2^{t+10}\equiv 2^t \pmod{22}$ for $t\ge 1$, it follows that

$$m \equiv \pm 4,\ \pm 6,\ \pm 10 \pmod{22}. \tag{11}$$

Note that, modulo 23, the sequence $\{Q_n\}$ is periodic with period 22. It follows from (11) and (3) that $Q_m \equiv Q_4, Q_6,$ or $Q_{10} \pmod{23}$. That is, $Q_m \equiv 17, 7,$ or $5 \pmod{23}$, so that

$$\left(\frac{Q_m}{23}\right) = \left(\frac{17}{23}\right), \left(\frac{7}{23}\right), \text{ or } \left(\frac{5}{23}\right)$$

and, in any case, we have $\left(\frac{Q_m}{23}\right) = -1$. This, together with (10) gives

$$\left(\frac{24Q_n+1}{Q_m}\right) = -1 \text{ for } n \notin \{0, 1\},$$

showing that $24Q_n+1$ is not a square. Hence the lemma.

Lemma 3: Suppose $n \equiv 3 \pmod{252}$. Then $24Q_n+1$ is a perfect square if and only if $n = 3$.

Proof: If $n = 3$, then $24Q_n+1 = 24 \cdot 7 + 1 = 13^2$. Conversely, if $n \equiv 3 \pmod{252}$ and $n \neq 3$, then we can write n as $n = 2 \cdot 3^2 \cdot 7 \cdot 2^r \cdot g + 3$, where $r \geq 1$ and g is odd. Writing

$$k = \begin{cases} 7 \cdot 3 \cdot 2^r & \text{if } r \equiv 11 \text{ or } 52 \pmod{82}, \\ 7 \cdot 2^r & \text{if } r \equiv 21, 26, 31, \text{ or } 67 \pmod{82}, \\ 3^2 \cdot 2^r & \text{if } r \equiv 1, 4, 16, 17, 20, 28, 33, 42, 45, \\ & \quad 57, 58, 61, 69, \text{ or } 74 \pmod{82}, \\ 3 \cdot 2^r & \text{if } r \equiv 3, 5, \pm 6, 7, \pm 10, 12, \pm 18, 19, \pm 23, 32, \\ & \quad \pm 35, 44, 46, 48, 51, 53, 60, \text{ or } 73 \pmod{82}, \\ 2^r & \text{otherwise}, \end{cases}$$

we find that $n = 2km+3$, where k is odd (in fact, $k = 3 \cdot g$, $3^2 \cdot g$, $7 \cdot g$, $3 \cdot 7g$, or $3^2 \cdot 7 \cdot g$). Therefore, by Lemma 1 and the facts that $Q_3 = 7$, k is odd, we have

$$24Q_n + 1 = 24P_{2km+3} + 1 \equiv 24(-1)^k Q_3 + 1 \pmod{Q_m} \equiv -167 \pmod{Q_m}.$$

Hence,

$$\left(\frac{24Q_n+1}{Q_m}\right) = \left(\frac{-167}{Q_m}\right) = \left(\frac{-1}{Q_m}\right)\left(\frac{167}{Q_m}\right) = \left(\frac{-1}{Q_m}\right)\left(\frac{Q_m}{167}\right)\left(\frac{-1}{Q_m}\right) = \left(\frac{Q_m}{167}\right). \tag{12}$$

Since $2^{t+82} \equiv 2^t \pmod{166}$ for $t \geq 1$, it follows that

$$\begin{aligned} m \equiv &\pm 4, \pm 14, \pm 18, \pm 20, \pm 22, \pm 24, \pm 26, \pm 40, \pm 42, \pm 50, \pm 52, \pm 58, \\ &\pm 62, \pm 66, \pm 70, \pm 72, \pm 74, \pm 76, \pm 78, \text{ or } \pm 82 \pmod{166}. \end{aligned} \tag{13}$$

But, modulo 167, the sequence $\{Q_n\}$ has period 166. This, together with (13) and (3), gives that

$$\begin{aligned} Q_m \equiv &\ 17, 15, 153, 55, 10, 5, 20, 37, 95, 30, 131, 123, 86, \\ &129, 125, 151, 113, 26, 43, \text{ or } 13 \pmod{167} \end{aligned}$$

and it can be seen that $\left(\frac{Q_m}{167}\right) = -1$ in all cases. Using this in (12), we get $\left(\frac{24Q_n+1}{Q_m}\right) = -1$, proving the theorem.

A consequence of Lemmas 2 and 3 is the following.

Lemma 4: Suppose $n \equiv 0$, 1, or 3 (mod 2520). Then $24Q_n+1$ is a perfect square only for $n = 0$, 1, or 3.

Lemma 5: $24Q_n+1$ is not a perfect square if $n \not\equiv 0$, 1, or 3 (mod 2520).

Proof: We prove the lemma in different steps, eliminating at each stage certain integers n modulo 2520 for which $24Q_n+1$ is not a square. In each step, we choose an integer m such that the period k (of the sequence $\{Q_n\}$ mod m) is a divisor of 2520 and thereby eliminate certain residue classes modulo k. Table A gives the various choices of the modulo m, the corresponding period k of Q_n modulo m, the values of n (mod k) for which the Jacobi symbol $(24Q_n+1/m)$ is -1 and the values of n (mod k) remaining at each stage. For example,

Modulo 7: The sequence $\{Q_n\}$ has period 6 so that, if $n \equiv 2$, 4, or 5 (mod 6), then $Q_n \equiv Q_2$, Q_4, or Q_5 (mod 7). Thus, we have $Q_n \equiv 3$ or 6 (mod 7); hence, $24Q_n+1 \equiv 3$ or 5 (mod 7). Therefore,

$$\left(\frac{24Q_n+1}{7}\right)=\left(\frac{3}{7}\right) \text{ or } \left(\frac{5}{7}\right),$$

showing that $(24Q_n+1/7)=-1$ and, hence, $24Q_n+1$ is not a square. Thus, $24Q_n+1$ is not a square if $n \equiv 2$, 4, or 5 (mod 6). So there remain the cases $n \equiv 0$, 1, or 3 (mod 6); equivalently, the cases $n \equiv 0$, 1, 3, 6, 7, or 9 (mod 12).

TABLE A

Period k	Modulus m	Values of n where $\left(\frac{24Q_n+1}{m}\right)=-1$	Left out values of n (mod t) where t is a positive integer
6	7	±2 and 5.	0, 1 or 3 (mod 6)
12	5	6, 7 and 9.	0, 1 or 3 (mod 12)
24	11	12 and 13.	0, 1, 3 or 15 (mod 24)
72	179	15, 25 and 51.	0, 1, 3, 27 or 63 (mod 72).
	73	±24, 39 and 49.	
504	1259	75, 99, 135, 145, 171, ±216, 217, 219, 243, 289, 351, 361, 433 and 459.	0, 1, 3, 63, 147 or 315 (mod 504)
56	337	11, ±16, 17, ±24, 39, 47 and 55.	
	113	31 and 51.	
42	4663	27.	
126	127	57.	
10	41	7.	0, 1 or 3 (mod 2520).
20	29	±4 and 15.	
30	31	9 and 19.	
60	269	25 and 51.	
40	19	33.	
	59	±8 and 23.	
280	139	203.	

We are now able to prove the following theorem.

Theorem 1: ***(a)*** Q_n is a generalized pentagonal number only for $n=0$, 1, or 3, and ***(b)*** Q_n is a pentagonal number only for $n=0$ or 1.

Proof: Part ***(a)*** of the theorem follows from Lemmas 4 and 5. For part ***(b)***, since an integer N is pentagonal if and only if $24N+1=(6m-1)^2$, where m is a positive integer, and since $Q_3=7$, we have $24Q_3+1 \neq (6m-1)^2$ for positive integer m, it follows that Q_3 is not pentagonal.

4. SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS

It is well known that, if $x_1 + y_1\sqrt{D}$ is the fundamental solution of Pell's equation $x^2 - Dy^2 = \pm 1$, where D is a positive integer which is not a perfect square, then $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$ is also a solution of the same equation; conversely, every solution of $x^2 - Dy^2 = \pm 1$ is of this form.

Now, by (5), we have $Q_n^2 = 2P_n^2 + (-1)^n$ for every n. Therefore, it follows that

$$Q_{2n} + \sqrt{2}P_{2n} \text{ is a solution of } x^2 - 2y^2 = 1, \tag{14}$$

while

$$Q_{2n+1} + \sqrt{2}P_{2n+1} \text{ is a solution of } x^2 - 2y^2 = -1. \tag{15}$$

Theorem 2: The solution set of the Diophantine equation

$$x^2(3x-1)^2 = 8y^2 + 4 \tag{16}$$

is $\{(1, 0)\}$.

Proof: Writing $X = x(3x-1)/2$, equation (16) reduces to the form

$$X^2 - 2y^2 = 1, \tag{17}$$

whose solutions are, by (14), $Q_{2n} + \sqrt{2}P_{2n}$ for any integer n.

Now $x = a$, $y = b$ is a solution of (16) $\Leftrightarrow$ $\{a(3a-1)/2\} + \sqrt{2}b$ is a solution of (17) $\Leftrightarrow$ $a(3a-1)/2 = Q_{2n}$ and $b = P_{2n}$ for some integer n.

Therefore, by Theorem 1(a), the ordered pair $\left(\frac{a(3a-1)}{2}, b\right) = (Q_0, P_0)$, giving that $(a, b) = (1, 0)$ and proving the theorem.

Similarly, we can prove the following theorem.

Theorem 3: The solution set of the Diophantine equation

$$x^2(3x-1)^2 = 8y^2 - 4 \tag{18}$$

is $\{(1, \pm 1), (-2, \pm 5)\}$.

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AMS Classification Numbers: 11B39, 11D25, 11B37

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