ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-577 ***Proposed by Paul S. Bruckman, Sacramento, CA***

Define the following constant: $C \equiv \prod_p \{1 - 1/p(p-1)\}$ as an infinite product over all primes p.

(A) Show that

$$\sum_{n=1}^{\infty} \mu(n)/\{n\phi(n)\},$$

where $\mu(n)$ and $\phi(n)$ are the Möbius and Euler functions, respectively.

(B) Let

$$P(n) = \sum_{d|n} \mu(n/d) L_d.$$

It was shown in the solution to H-517 (see Vol. 35, no. 4 (1997), pp. 381-82) that $n|P(n)$.

Show that

$$C = \prod_{n=2}^{\infty} \{\zeta(n)\}^{-P(n)/n},$$

where ζ is the Riemann zeta function.

Note: C is the conjectured density of primes p such that $Z(p) = p - (5/p)$; see P. G. Anderson & P. S. Bruckman, "On the a-Densities of the Fibonacci Sequence," *NNTDM* **6.1** (2000):1-13. Approximately, $C = 0.37395581$.

H-578 ***Proposed by N. Gauthier & J. R. Gosselin, Royal Military College of Canada***

In Problem B-863, S. Rabinowitz gave a set of four 2×2 matrices which are particular solutions of the matrix equation

$$X^2 = X + I, \tag{1}$$

where I is the unit matrix [*The Fibonacci Quarterly* **36.5** (1998); solved by H. Kappus, **37.3** (1999)]. The matrices presented by Rabinowitz are not diagonal (i.e., they are nontrivial), have determinant -1 and trace $+1$.

a. Find the complete set $\{X\}$ of the nontrivial solutions of (1) and establish whether the properties $\det(X) = -1$ and $\mathrm{tr}(X) = +1$ hold generally.

b. Determine the complete set $\{X\}$ of the nontrivial solutions of the generalized characteristic equation

$$X^2 = xX + yI, \tag{2}$$

for the 2×2 Fibonacci matrix sequence $X^{n+2} = xX^{n+1} + yX^n$, $n = 0, 1, 2, \ldots$, where x and y are arbitrary parameters such that $x^2/4 + y \neq 0$; obtain expressions for the determinant and for the trace.

H-579 ***Proposed by Paul S. Bruckman, Sacramento, CA***

Prove or disprove that, for all odd primes p,

$$\sum_{n=1}^{1/2(p-1)} \binom{2n}{n}(-1)^n / n \equiv 0 \pmod{p}.$$

Each quantity $1/n$ is to be interpreted as $n^{-1} \pmod{p}$.

H-580 ***Proposed by José Díaz-Barrero, Politechnic University of Cataluya, Barcelona, Spain***

Let

$$A(z) = \sum_{k=0}^{n} a_k z^k$$

be a monic complex polynomial. Show that all its zeros lie in the disk $C = \{z \in \mathbb{C} : |z| < r\}$, where

$$r = \max_{1 \le k \le n} \left\{ \sqrt[k]{\frac{L_{j+3n}}{C(n,k) 2^k L_{j+k}} |a_{n-k}|} \right\},\ j = 0, 1, 2, \ldots.$$

SOLUTIONS

Sum Problem

H-566 ***Proposed by N. Gauthier, Royal Military College of Canada***
(Vol. 38, no. 4, August 2000)

Let $\phi_n := \pi/2n$, where n is a positive integer, and set $L_n = a^n + b^n$, $F_n = (a^n - b^n)/(a-b)$, where $a = \frac{1}{2}(u + \sqrt{u^2-4})$, $b = \frac{1}{2}(u - \sqrt{u^2-4})$, $u \neq \pm 2$, and show that, for $n \ge 2$,

$$S_n(u) := \sum_{k=1}^{n-1} \frac{1}{1 + \left(\frac{u+2}{u-2}\right) tg^2(k\phi_n)} = -\frac{1}{2} + \frac{n}{2(u+2)^2 F_n}[L_{n+1} + 3L_n + 3L_{n-1} + L_{n-2}].$$

Solution by Paul S. Bruckman, Berkeley, CA

The notation "tg^2" in the statement of the problem evidently means "$\tan^2$". We make use of a general identity which was derived in this solver's solution of Problem H-559, Part (a) (Feb. 2000) by the proposer of this problem. This identity is the following, valid for all integers $n \ge 1$, complex x and y with $x^2 \neq y^2$:

$$\frac{n(x^n + y^n)}{(x^2 - y^2)(x^n - y^n)} = \sum_{k=1}^{n} (x^2 - 2xy\cos 2k\pi/n + y^2)^{-1}. \tag{1}$$

If we set $x = a$ and $y = b$ in (1), we obtain (using the proposer's notation):

$$nL_n / \{u(u^2-4)F_n\} = \sum_{k=1}^{n} (u^2 - 2 - 2\cos 2k\pi/n)^{-1}. \tag{2}$$

We may transform the sum in the right member of (2) as follows:

$$u^2-2-2\cos 2k\pi/n = u^2-4\cos^2 k\pi/n,$$

so the sum becomes:

$$1/2u\sum_{k=1}^{n}\{(u-2\cos k\pi/n)^{-1}+(u+2\cos k\pi/n)^{-1}\}.$$

Therefore,

$$\begin{aligned}2nL_n/F_n &= (u^2-4)\sum_{k=1}^{n}\{(u-2\cos k\pi/n)^{-1}+(u+2\cos k\pi/n)^{-1}\}\\ &= (u^2-4)\sum_{k=1}^{n}\{(u+2-4\cos^2 k\phi_n)^{-1}+(u-2+4\cos^2 k\phi_n)^{-1}\}.\end{aligned}$$

Let D_k denote $1+\{(u+2)/(u-2)\}\tan^2 k\phi_n$. We note that the transformation $k\to n-k$ transforms $\tan k\phi_n$ to $1/\tan k\phi_n$. Then using standard trigonometric manipulations, we obtain after some effort:

$$\begin{aligned}2nL_n/F_n &= \sum_{k=1}^{n}\{u-2+4/D_k\}+\sum_{k=0}^{n-1}\{u+2-4(u+2)/(u-2)\tan^2 k\phi_n/D_k\}\\ &= (u-2)n+\sum_{k=1}^{n-1}4/D_k+(u+2)n-4\sum_{k=1}^{n-1}(D_k-1)/D_k,\end{aligned}$$

since the terms for $k=0$ and $k=n$ involving D_k vanish. Then $nL_n/F_n = un-2(n-1)+4S_n(u)$. Hence, $4S_n(u)=nL_n/F_n-(u-2)n-2$, or

$$S_n(u) = -1/2+n\{L_n-(u-2)F_n\}/4F_n. \tag{3}$$

We note that $a^2=au-1$, $b^2=bu-1$, which implies $(a+1)^2=a(u+2)$, $(b+1)^2=b(u+2)$, so $(a+1)^3=a(a+1)(u+2)$, $(b+1)^3=b(b+1)(u+2)$. The sum $L_{n+1}+3L_n+3L_{n-1}+L_{n-2}$ reduces to:

$$\begin{aligned}a^{n-2}(a^3+3a^2+3a+1)+b^{n-2}(b^3+3b^2+3b+1) &= a^{n-2}(a+1)^3+b^{n-2}(b+1)^3\\ &= a^{n-1}(a+1)(u+2)+b^{n-1}(b+1)(u+2)=(u+2)(L_n+L_{n-1}).\end{aligned}$$

Therefore, letting $R_n(u)$ denote the expression in the right member of the statement of the problem, we have

$$R_n(u) = n(L_n+L_{n-1})/\{2(u+2)F_n\}-1/2. \tag{4}$$

We next prove the following identity:

$$uL_n = 2L_{n-1}+(u^2-4)F_n. \tag{5}$$

Proof of (5): Let $\theta=(u^2-4)^{1/2}$. Note that $a-b=\theta$, $ab=1$. Hence,

$$\begin{aligned}2L_{n-1}+(u^2-4)F_n &= a^{n-1}(2+\theta a)+b^{n-1}(2-\theta b)\\ &= a^{n-1}(1+a^2)+b^{n-1}(1+b^2)=L_{n+1}+L_{n-1}=uL_n,\end{aligned}$$

since the characteristic equation of the L_n's (and of the F_n's) is $U_{n+1}=uU_n-U_{n-1}$. □

Comparing the results of (3) and (4), we see that it suffices to prove the following:

$$(L_n+L_{n-1})/(u+2) = (L_n-(u-2)F_n\}/2,$$

which (after simplification) reduces to (5). Thus, $S_n(u) = R_n(u)$, which completes the proof.

Incidentally, although not required, we could also express the result as follows:

$$S_n(u) = a_n - a_1, \text{ where } a_n = n(F_n - F_{n-1}) / 2F_n. \tag{6}$$

Also solved by H.-J. Seiffert and the proposer.

An UnEqual Problem

H-567 ***Proposed by Ernst Herrmann, Siegburg, Germany***
(Vol. 38, no. 5, November 2000)

Let F_n denote the n^{th} Fibonacci number. For any natural number $n \geq 3$, the four inequalities

$$\begin{aligned}\frac{1}{F_n} + \frac{1}{F_{n+a_1}} &< \frac{1}{F_{n-1}} \\ &\leq \frac{1}{F_n} + \frac{1}{F_{n+a_1-1}},\end{aligned} \tag{1}$$

$$\begin{aligned}\frac{1}{F_n} + \frac{1}{F_{n+a_1}} + \frac{1}{F_{n+a_1+a_2}} &< \frac{1}{F_{n-1}} \\ &\leq \frac{1}{F_n} + \frac{1}{F_{n+a_1}} + \frac{1}{F_{n+a_1+a_2-1}},\end{aligned} \tag{2}$$

determine uniquely two natural numbers a_1 and a_2. Find the numbers a_1 and a_2 dependent on n.

Solution by H.-J. Seiffert, Berlin, Germany

It is known [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials," *The Fibonacci Quarterly* **23.1** (1985):7-20, Identity (3.32)] that

$$F_{m+h}F_{m+k} - F_m F_{m+h+k} = (-1)^m F_h F_k, \ m, h, k \in \mathbf{Z}.$$

With $(m, h, k) = (n, -1, 1)$, $(n-1, 1, 2)$, $(n-1, -1, 4)$, $(n-1, -1, 5)$, $(n-1, -1, 2)$, $(n, -2, 2)$, we obtain, respectively:

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \tag{3}$$

$$F_n F_{n+1} - F_{n-1}F_{n+2} = (-1)^{n-1}, \tag{4}$$

$$F_{n-2}F_{n+3} - F_{n-1}F_{n+2} = 3(-1)^{n-1}, \tag{5}$$

$$F_{n-2}F_{n+4} - F_{n-1}F_{n+3} = 5(-1)^{n-1}, \tag{6}$$

$$F_{n-2}F_{n+1} - F_{n-1}F_n = (-1)^{n-1}, \tag{7}$$

$$F_{n-2}F_{n+2} - F_n^2 = (-1)^{n-1}, \tag{8}$$

valid for all integers n. Also, the easily verifiable identity

$$F_{3n} = 5F_n^3 + 3(-1)^n F_n, \ n \in \mathbf{Z}, \tag{9}$$

and the following inequalities are needed below.

Lemma 1: For all integers $n \geq 3$, it holds that

$$F_{3n-4} < F_{n-1}F_n F_{n+1} < F_{3n-3}.$$

Proof: From (3) and (9), we have

$$F_{n-1}F_nF_{n+1} = F_n^3 + (-1)^n F_n = \frac{1}{5}(F_{3n} + 2(-1)^n F_n),$$

showing that the left side is equivalent to $5F_{3n-4} < F_{3n} + 2(-1)^n F_n$. But this inequality follows from $F_{3n} = 5F_{3n-4} + 3F_{3n-5}$ and, by $n \geq 3$, $F_{3n-5} > F_n$. Using (9) with n replaced by $n-1$, we see that the right side is equivalent to $F_nF_{n+1} < 5F_{n-1}^2 - 3(-1)^n$, so that, in view of (4), we must show that $F_{n-1}F_{n+2} < 5F_{n-1}^2 - 2(-1)^n$. Since $F_{n+2} = 3F_{n-1} + 2F_{n-2}$, this inequality becomes $F_{n-2}F_{n-1} < F_{n-1}^2 - (-1)^n$. Clearly, this holds for $n = 3$. For $n \geq 4$, the latter inequality follows from $1 \leq F_{n-2} \leq F_{n-1} - 1$. Q.E.D.

Lemma 2: For all integers $n \geq 5$, it holds that

$$F_{n+3} < \frac{F_{n-1}F_nF_{n+2}}{F_{n-2}F_n - 1} < F_{n+4}.$$

Proof: After multiplying the left side by $F_{n-2}F_n - 1 > 0$, we see that we must prove the inequality $F_n(F_{n-2}F_{n+3} - F_{n-1}F_{n+2}) < F_{n+3}$ which, by (5), reduces to $3(-1)^n F_n < F_{n+3}$. However, this follows from $F_{n+3} = 2F_{n+1} + F_n > 3F_n$. Similarly, the right side is equivalent to

$$F_{n+4} < F_n(F_{n-2}F_{n+4} - F_{n-1}F_{n+2}).$$

Using $F_{n+2} = F_{n+3} - F_{n+1}$ and (6), we see that we must prove that $F_{n+4} < 5(-1)^{n-1}F_n + F_{n-1}F_nF_{n+1}$, which follows from $F_{n+4} = 5F_n + 3F_{n-1} < 8F_n$ and $13 < F_{n-1}F_{n+1}$; note that $n \geq 5$. Q.E.D.

Let $n \geq 3$ be an integer. Since $F_n - F_{n-1} = F_{n-2}$, it is easily verified that (1) is equivalent to

$$F_{n-2}F_{n+a_1-1} \leq F_{n-1}F_n < F_{n-2}F_{n+a_1}.$$

We claim that

$$a_1 = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

For odd n, we must show that $F_{n-2}F_n \leq F_{n-1}F_n < F_{n-2}F_{n+1}$. The left side is obviously true, while the right side follows from (7). If n is even, we must show that $F_{n-2}F_{n+1} \leq F_{n-1}F_n < F_{n-2}F_{n+2}$, whose left side again follows from (7). To prove the right side, observe that $1 \leq F_{n-1} \leq F_n - 1$, so that $F_{n-2}F_{n+2} - F_{n-1}F_n \geq F_{n-2}F_{n+2} - F_n^2 + F_n = F_n - 1 > 0$, where we have used (8). This proves the above claim. Next, we shall prove that

$$a_2 = \begin{cases} 2n-4 & \text{if n is odd,} \\ 3 & \text{if } n = 4, \\ 2 & \text{if } n > 4 \text{ is even.} \end{cases}$$

If $n \geq 3$ is odd, then $a_1 = 1$, and based on $F_n + F_{n+1} = F_{n+2}$ and (4), it is easily seen that (2) becomes $F_{n+a_2} \leq F_{n-1}F_nF_{n+1} < F_{n+a_2+1}$. Applying Lemma 1, it follows that $a_2 = 2n-4$. If $n \geq 4$ is even, then $a_1 = 2$ and, since $F_n - F_{n-1} = F_{n-2}$, (2) is equivalent to

$$F_{n+a_2+1} \leq \frac{F_{n-1}F_nF_{n+2}}{F_{n-2}F_{n+2} - F_{n-1}F_n} < F_{n+a_2+2}.$$

Since, by (8), $F_{n-2}F_{n+2} = F_n^2 - 1$, and since $F_n - F_{n-1} = F_{n-2}$, these inequalities are equivalent to

$$F_{n+a_2+1} \leq \frac{F_{n-1}F_nF_{n+2}}{F_{n-2}F_n - 1} < F_{n+a_2+2}.$$

Direct computation shows that $a_2 = 3$ if $n = 4$. For even $n \geq 6$, from Lemma 2, we find $a_2 = 2$. This completes the solution.

Also solved by P. Bruckman, L. A. G. Dresel, R. Martin, and the proposer.

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Announcement

TENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

June 24–June 28, 2002

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LOCAL INFORMATION

For information on local housing, food, tours, etc., please contact:

Professor Calvin T. Long
2120 North Timberline Road
Flagstaff, AZ 86004
e-mail: calvin.long@nau.edu Fax: 928-523-5847 Phone: 928-527-4466

CALL FOR PAPERS

The purpose of the conference is to bring together people from all branches of mathematics and science who are interested in Fibonacci numbers, their applications and generalizations, and other special number sequences. For the conference *Proceedings*, manuscripts that include new, unpublished results (or new proofs of known theorems) will be considered. A manuscript should contain an abstract on a separate page. For papers not intended for the *Proceedings*, authors may submit just an abstract, describing new work, published work or work in progress. Papers and abstracts, which should be submitted in duplicate to F. T. Howard at the address below, are due no later than May 1, 2002. Authors of accepted submissions will be allotted twenty minutes on the conference program. Questions about the conference may be directed to:

Professor F. T. Howard
Wake Forest University
Box 7388 Reynolda Station
Winston-Salem, NC 27109 (U.S.A.)
e-mail: howard@mthcsc.wfu.edu