ENUMERATION OF PATHS, COMPOSITIONS OF INTEGERS, AND FIBONACCI NUMBERS

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1. INTRODUCTION

We study certain paths in the first quadrant from $(0,0)$ to (i,j). These paths consist of segments from $(x_0, y_0)=(0,0)$ to (x_1, y_1) to ... to $(x_k, y_k)=(i,j)$. If we write

$$p_h = x_h - x_{h-1},\ q_h = y_h - y_{h-1},\ h=1,2,\dots,k,$$

then $p_h \ge 0$, $q_h \ge 0$, and

$$p_1+p_2+\cdots+p_k = i,\ q_1+q_2+\cdots+q_k = j.$$

Thus, the p_h form a composition of i, the q_h a composition of j.

The paths we shall study will have some restrictions on the p_h and q_h. For example, in Section 3, we shall enumerate paths for which

$$a_x \le p_h \le b_x,\ a_y \le q_h \le b_y,\ h=1,2,\dots,k,\ k\ge 1,$$

where $a_x \ge 1$, $a_y \ge 1$.

2. COMPOSITIONS

A *composition* of a nonnegative integer n is a vector $(p_1, \dots, p_k)$ for which

$$p_1+p_2+\cdots+p_k = n,\ p_h \ge 0.$$

Note that the order in which the p_h are listed matters. Each p_h is called a *part*, and k, the *number of parts*. Let $c(n,k,a,b)$ be the number of compositions of n into k parts p_h with $a \le p_h \le b$. It is well known that

$$c(n,k,0,\infty) = \binom{n+k-1}{k-1}.$$

On subtracting a from each part, it is easy to see that

$$c(n,k,a,\infty) = c(n-ka,k,0,\infty) = \binom{n-ka+k-1}{k-1}. \tag{1}$$

With $a=1$, this gives the number of compositions of n into k positive parts,

$$c(n,k,1,\infty) = \binom{n-1}{k-1},$$

and the number of compositions of n into positive parts,

$$\sum_{k=1}^{n} c(n,k,1,\infty) = \sum_{k=1}^{n}\binom{n-1}{k-1} = 2^{n-1}.$$

3. SOME RESULTS ON PATHS

Theorem 1: Suppose a_x, b_x, a_y, b_y are integers satisfying $1 \le a_x \le b_x$ and $1 \le a_y \le b_y$. Let $T(0, 0) = 1$, and for $i \ge 0$ and $j \ge 0$ but not $i = j = 0$, let $T(i, j)$ be the number of paths satisfying

$$a_x \le p_h \le b_x \quad \text{and} \quad a_y \le q_h \le b_y \tag{2}$$

for $h = 1, 2, \ldots, k$ and $k = 1, 2, \ldots, \min(i, j)$. Then

$$T(i, j) = \sum_{k=1}^{\min(i, j)} c(i, k, a_x, b_x) c(j, k, a_y, b_y). \tag{3}$$

Proof: Suppose $1 \le k \le \min(i, j)$. Corresponding to each path of the sort described in the introduction satisfying (2) is a composition of i into k parts and a composition of j into k parts satisfying (2), and conversely.

There are $c(i, k, a_x, b_x)$ such compositions of i and $c(j, k, a_y, b_y)$ such compositions of j, hence $c(i, k, a_x, b_x)c(j, k, a_y, b_y)$ such paths consisting of k segments. Summing over all possible numbers of segments yields (3). □

Corollary 1.1: The number of paths $T(i, j)$ with $p_h \ge a_x$, $q_h \ge a_y$ (where $a_x \ge 1$, $a_y \ge 1$) is

$$T(i, j) = \sum_{k=1}^{\min(i, j)} \binom{i - ka_x + k - 1}{k - 1} \binom{j - ka_y + k - 1}{k - 1}.$$

Proof: Let $b_x = \infty$ and $b_y = \infty$ in Theorem 1. □

Lemma 1.1: If I and J are nonnegative integers, then

$$\sum_{k=0}^{I} \binom{I}{k} \binom{J}{k} = \binom{I + J}{I} \quad \text{and} \quad \sum_{k=1}^{I} \binom{I}{k} \binom{J}{k - 1} = \binom{I + J}{I - 1}.$$

Proof: The assertions clearly hold for $I = J = 0$. Suppose that $I + J \ge 1$ and that both identities hold for all I' and J' satisfying $I' + J' < I + J$. Then

$$\sum_{k=0}^{I} \binom{I}{k} \binom{J}{k} = \sum_{k=0}^{I} \binom{I}{k} \binom{J-1}{k-1} + \sum_{k=0}^{I} \binom{I}{k} \binom{J-1}{k}$$
$$= \binom{I + J - 1}{I - 1} + \binom{I + J - 1}{I} = \binom{I + J}{I}$$

and

$$\sum_{k=1}^{I} \binom{I}{k} \binom{J}{k-1} = \sum_{k=1}^{I} \binom{I-1}{k-1} \binom{J}{k-1} + \sum_{k=1}^{I} \binom{I-1}{k} \binom{J}{k-1}$$
$$= \binom{I + J - 1}{I - 1} + \binom{I + J - 1}{I - 2} = \binom{I + J}{I - 1}.$$

Although both of the identities in Lemma 1.1 are needed inductively in the foregoing proof, only the first identity will be used below.

Corollary 1.2: The number $T(i, j)$ of paths satisfying $p_h \ge 1$, $q_h \ge 1$ is given by

$$T(i, j) = \binom{i+j-2}{i-1}.$$

Proof: Put $a_h = 1$, $a_y = 1$ in Corollary 1.1, and apply Lemma 1.1. □

Theorem 2: The number $T(i, j)$ of paths satisfying $p_h \geq 1$, $q_h \geq 0$ is given by

$$T(i, j) = \sum_{k=1}^{i} c(i, k, 1, \infty)c(j, k, 0, \infty) = \sum_{k=1}^{i} \binom{i-1}{k-1}\binom{j+k-1}{k-1}.$$

Proof: The method of proof is essentially the same as for Theorem 1. Here, however, the greatest k for which there is a path for which all $p_h \geq 1$ is i, rather than $\min(i, j)$. □

The array of Theorem 2 is of particular interest; for example:

(A) $T(i, 0) = 2^{i-1}$ for $i \geq 1$;

(B) $T(i, 1) = (i+1)2^{i-2}$ for $i \geq 1$;

(C) $\sum_{\substack{i+j=n \\ i\geq 0,\, j\geq 0}} T(i, j) = F_{2n}$ for $n \geq 1$, i.e., antidiagonal sums are Fibonacci numbers;

(D) $\sum_{\substack{i+j=n \\ i\geq 0,\, j\geq 0}} (-1)^j T(i, j) = F_n$ for $n \geq 1$, i.e., alternating antidiagonal sums are Fibonacci numbers;

(E) the diagonal $T(n, n-1) = (1, 3, 13, 63, 321, \ldots)$ is the Delannoy sequence, A001850 in [1];

(F) the diagonal $T(n, n) = (1, 1, 4, 19, 96, 501, \ldots)$ is the sequence A047781 in [1].

We leave proofs of (A)-(F) to the reader, along with the determination of the position and magnitude of the maximum number M_n in the n^{th} antidiagonal of T. The first fourteen values of M_n are 1, 2, 4, 8, 20, 48, 112, 272, 688, 1696, 4096, 10496, 26624, 66304. One wonders what can be said about $\lim_{n\to\infty} M_n / F_{2n}$. Initial terms of the sequences in (A)-(F) appear in Figure 1.

The $T(i, j)$ given in Theorem 2 are determined recursively by $T(0, 0) = 1$, $T(0, j) = 0$ for $j \geq 1$, $T(i, 0) = 2^{i-1}$ for $i \geq 1$, and

$$T(i, j) = \sum_{k=0}^{i-1} \sum_{l=0}^{j} T(k, l). \tag{4}$$

To verify (4), note that each path with final segment terminating on (i, j) has penultimate segment terminating on a lattice point (h, s) in the rectangle

$$R_{i,j} := \{0, 1, \ldots, i-1\} \times \{0, 1, \ldots, j\}.$$

Therefore, the number of relevant paths from $(0, 0)$ to (i, j) is the sum of the numbers of such paths from $(0, 0)$ to a point in $R_{i,j}$.

More generally, all arrays as in Theorems 1 and 2 are determined recursively by

$$T(i, j) = \sum_{(h,s)\in R_{i,j}} T(h, s) \text{ for } i \geq 2 \text{ and } j \geq 1,$$

where initial values and lattice point sets $R_{i,j}$ are determined by (2) or other conditions.

0	1	10	64	328	1462	5908	22180
0	1	9	53	253	1059	4043	14407
0	1	8	43	190	743	2668	8989
0	1	7	34	138	501	1683	5336
0	1	6	26	96	321	1002	2972
0	1	5	19	63	192	552	1520
0	1	4	13	38	104	272	688
0	1	3	8	20	48	112	256
1	1	2	4	8	16	32	64

FIGURE 1. $p_h \geq 1$, $q_h \geq 0$ **(see Theorem 2)**

4. RESTRICTED HORIZONTAL COMPONENTS

Suppose $m \geq 2$. In this section, we determine $T(i, j)$ when $1 \leq p_h \leq m$ and $q_h \geq 0$. By the argument of Theorem 2,

$$T(i, j) = \sum_{k=1}^{i} c(i, k, 1, m)c(j, k, 0, \infty) = \sum_{k=1}^{i} c(i, k, 1, m)\binom{j+k-1}{k-1},$$

where $c(i, k, 1, m)$ is the number of compositions of i into k positive parts, all $\leq m$. By the remarks at the end of Section 3, the numbers $T(i, j)$ are determined recursively by $T(0, 0) = 1$, $T(0, j) = 0$ for $j \geq 1$, $T(i, 0) = 2^{i-1}$ for $i = 1, 2, \ldots, m$, and

$$T(i, j) = \sum_{k=i-m}^{i-1} \sum_{l=0}^{j} T(k, l). \tag{5}$$

Values of $T(i, j)$ for $m = 2$ are shown in Figure 2.

In Figure 2, the antidiagonal sums $(1, 1, 3, 7, 17, 41, 99, \ldots)$ comprise a sequence that appears in many guises, such as the numerators of the continued-fraction convergents to $\sqrt{2}$, (See the sequence A001333 in [1].)

Also in Figure 2, the numbers in the bottom row, $T(i, 0)$, are the Fibonacci numbers. Since the other rows are easily obtained via (5) from these, it is natural to inquire about the bottom row when $m \geq 3$; we shall see, as a corollary to Theorem 3, that the m-Fibonacci numbers then occupy the bottom row.

0	1	10	63	309	1290	4797	16335
0	1	9	52	236	918	3198	10248
0	1	8	42	175	630	2044	6132
0	1	7	33	125	413	1239	3458
0	1	6	25	85	255	701	1806
0	1	5	18	54	145	361	850
0	1	4	12	31	73	162	344
0	1	3	7	15	30	58	109
1	1	2	3	5	8	13	21

FIGURE 2. Enumeration of Paths Consisting of Segments with Horizontal Components of Lengths 1 or 2

Lemma 3.1: Suppose $m \geq 1$. The number $F(m,n) = \sum_{k=1}^{n} c(n,k,1,m)$ of compositions of n into positive parts $\leq m$ is given by:

$$F(1,n) = 1 \text{ for } n \geq 1;$$

$$F(m,n) = 2^{n-1} \text{ for } 1 \leq n \leq m,\ m \geq 2;$$

$$F(m,n) = F(m,n-1) + F(m,n-2) + \cdots + F(m,n-m) \text{ for } n \geq m+1 \geq 3.$$

Proof: For row 1 of the array F, there is only one composition of n into positive parts all ≤ 1, namely, the n-dimensional vector $(1,1,\ldots,1)$, so that $F(1,n)=1$ for $n \geq 1$. Now suppose that the row number m is ≥ 2 and $1 \leq n \leq m$. Then every composition of n has all parts $\leq m$, and $F(m,n) = 2^{n-1}$.

Finally, suppose $n \geq m+1 \geq 3$. Each composition $(p_1, p_2, \ldots, p_k)$ of n into parts all $\leq m$ has a final part p_k that will serve our purposes. For $u = 1, 2, \ldots, m$, the set

$$S_u = \left\{(p_1, p_2, \ldots, p_k) : 1 \leq p_i \leq m \text{ for } i = 1, 2, \ldots, k;\ \sum_{i=1}^{k} p_i = n;\ p_k = u\right\}$$

is an obvious one-to-one correspondence with the set of compositions $(p_1, p_2, \ldots, p_{k-1})$ for which $1 \leq p_i \leq m$ for $i = 1, 2, \ldots, k-1$ and $\sum_{i=1}^{k-1} p_i = n-u$, of which, by the induction hypothesis, there are $F(m, n-u)$. The sets $S_1, S_2, \ldots, S_m$ partition the set of compositions to be enumerated, so that the total count is $F(m,n-1) + F(m,n-2) + \cdots + F(m,n-m)$. □

Theorem 3: Suppose $m \geq 1$. Then the bottom row of array T is given by $T(i, 0) = F(m, i+1)$ for $i \geq 0$.

Proof: We have $T(0, 0) = F(m, 1) = 1$. Suppose now that $i \geq 1$. The sum in (5) and initial values given with (5) yield

$$T(i, 0) = \begin{cases} 2^{i-1} & \text{if } 1 \leq i \leq m, \\ T(i-m, 0) + T(i-m+1, 0) + \cdots + T(i-1, 0) & \text{if } i \geq m+1, \end{cases}$$

and by Lemma 3.1,

$$F(m, i+1) = \begin{cases} 2^{i} & \text{if } 0 \leq i \leq m-1, \\ F(m, i) + F(m, i-1) + \cdots + F(m, i+1-m) & \text{if } i \geq m. \end{cases}$$

Thus, the initial values and recurrences of the sequences $\{T(i, 0)\}$ and $\{F(m, i+1)\}$ are identical, so that the sequences are equal.

REFERENCE

1. Neil J. A. Sloane. *Online Encyclopedia of Integer Sequences*. http://www.research.att.com/~njas/sequences/

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