SOME GENERAL FORMULAS ASSOCIATED WITH THE SECOND-ORDER HOMOGENEOUS POLYNOMIAL LINE-SEQUENCES

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(Submitted August 1999-Final Revision November 1999)

In our previous report [5], we developed some methods in the study of the line-sequential properties of the polynomial sequences treated in Shannon and Horadam [9]. In this report, we work out the properties for the general case and apply them to the Pell polynomial line-sequence as an example. Some known results are included for completeness, but only new aspects will be presented in some detail.

1. THE BASIC FORMULAS

Recall that the linear homogeneous second-order recurrence relation is given by

$$cu_n + bu_{n+1} = u_{n+2},\ c, b \neq 0,\ n \in \mathbf{Z}; \tag{1.0a}$$

and a general line-sequence is expressed as

$$\bigcup_{u_0,\, u_1}(c, b): \ldots, u_{-3}, u_{-2}, u_{-1}, [u_0, u_1], u_2, u_3, u_4, \ldots, u_n \in \mathbf{R}, \tag{1.0b}$$

where $[u_0, u_1]$ denotes the generating pair of the line-sequence.

1. Basis Pair. The basis pair for the general case, that is, without specifying the parametric coefficients b and c, is given by (4.2) and (4.3) in [8]:

$$G_{1,0}(c, b): \ldots, (c+b^2)/c^2, -b/c, [1, 0], c, cb, c(c+b^2), \ldots, \tag{1.1a}$$

$$G_{0,1}(c, b): \ldots, (c+b^2)/c^3, -b/c^2, 1/c, [0, 1], b, (c+b^2), \ldots. \tag{1.1b}$$

Definition 1: Two line-sequences are said to be *complementary* if they are orthogonal.

Obviously, the pair (1.1a) and (1.1b) are orthogonal and form a set of basis. When $c = b = 1$, they reduce to the complementary Fibonacci and the Fibonacci line-sequences, $F_{1,0}$ and $F_{0,1}$, respectively. It is clear that all the line-sequential properties of either a number line-sequence or a polynomial line-sequence given by the recurrence relation (1.0a) originate from the properties of this pair. Following are some of the main properties.

2. Translation. The translational relation between the basic pair is given by:

$$TG_{1,0} = cG_{0,1}, \tag{1.2a}$$

where T denotes the translation operation, see (3.1) in [8]. Let $g_n[i, j]$ denote the n^{th} term in the line-sequence $G_{i,j}$, then, in terms of the elements, (1.2a) becomes

$$g_{n+1}[1, 0] = cg_n[0, 1]. \tag{1.2b}$$

3. Parity. The parity relation of the elements in $G_{1,0}$ is found to be

$$g_{-n}[1, 0] = (-1)^n c^{-(n+1)} g_{n+2}[1, 0]. \tag{1.3a}$$

From (4.9) in [8], the parity relation of the elements in $G_{0,1}$ is given by

$$g_{-n}[0,1]=(-1)^{n+1}c^{-n}g_n[0,1]. \tag{1.3b}$$

Applying translational relation (1.2b) to (1.3b), we get (1.3a). In the nomenclature of Shannon and Horadam [9], parity relation (1.3b) reduces to (1.7) in [1] for $c=-1$ in the case of Morgan-Voyce even Fibonacci polynomials.

4. Cross Relations. Combining the translational and parity relations, we obtain the following set of *cross* relations among the elements of the two basis polynomial line-sequences:

$$g_{-n}[1,0]=(-1)^n c^{-n}g_{n+1}[0,1], \tag{1.4a}$$

$$g_{-n}[1,0]=cg_{-(n+1)}[0,1]; \tag{1.4b}$$

or

$$g_{-n}[0,1]=(-1)^{n+1}c^{-(n+1)}g_{n+1}[1,0], \tag{1.4c}$$

$$g_{-n}[0,1]=c^{-1}g_{-(n-1)}[1,0]. \tag{1.4d}$$

5. Geometrical Line-Sequences. The pair of geometrical line-sequences relating to $G_{1,0}$ is given by:

$$G_{1,\alpha}(c,b):\ldots,\alpha^{-2},\alpha^{-1},[1,\alpha],\alpha^2,\alpha^3,\ldots, \tag{1.5a}$$

$$G_{1,\beta}(c,b):\ldots,\beta^{-2},\beta^{-1},[1,\beta],\beta^2,\beta^3,\ldots, \tag{1.5b}$$

where α and β are the roots of the generating equation

$$q^2-bq-c=0 \tag{1.5c}$$

(ref. (1.7) in [4], with $g=0$).

6. Binet's Formula. Binet's formula for the $G_{1,0}$ basis is given by

$$G_{1,0}=(-\beta G_{1,\alpha}+\alpha G_{1,\beta})/(\alpha-\beta), \tag{1.6a}$$

and for the $G_{0,1}$ basis is given by

$$G_{0,1}=(G_{1,\alpha}-G_{1,\beta})/(\alpha-\beta) \tag{1.6b}$$

(ref. (4.9) in [7]).

7. (General) Lucas Pair. Recall that the line-sequence "conjugate" to $G_{0,1}$ is the "general" Lucas line-sequence generated by $[2,b]$, see (4.12) in [8]:

$$G_{2,b}(c,b):\ldots,(2c+b^2)/c^2,-b/c,[2,b],2c+b^2,b(3c+b^2),\ldots, \tag{1.7a}$$

which reduces to the Lucas line-sequence $L_{2,1}$ if $c=b=1$. Its complement is

$$G_{b,-2}(c,b):\ldots,b(c+2+b^2)/c^2,-(2+b^2)/c,[b,-2],(c-2)b,-2c+(c-2)b^2,\ldots. \tag{1.7b}$$

For $c=b=1$, it reduces to the complementary Lucas line-sequence,

$$L_{1,-2}(1,1):\ldots,-7,4,-3,[1,-2],-1,-3,-4,-7,\ldots. \tag{1.7c}$$

The orthogonal pair (1.7a) and (1.7b) then form a "(general) Lucas basis" spanning the same $2D$ space as does the basis pair $G_{1,0}$ and $G_{0,1}$, but with a normalization factor $(b^2+4)^{-1/2}$.

8. Decomposition Schemes. Thus, we have the following different ways of decomposing a given line-sequence, $G_{i,j}(c,b)$: The first is the "basic decomposition" resulting in the basic component expression, see (2.9) in [8]:

$$G_{i,j}(c,b) = iG_{1,0}(c,b) + jG_{0,1}(c,b). \tag{1.8a}$$

The second is the "Binet decomposition" the general formula of which is

$$G_{i,j}(c,b) = [-(\beta i - j)G_{1,\alpha} + (\alpha i - j)G_{1,\beta}]/(\alpha - \beta). \tag{1.8b}$$

Note that the Binet pair $G_{1,\alpha}$ and $G_{1,\beta}$ does not form an orthogonal pair unless $c=1$, see (2.8) in [8].

The third is the "Lucas decomposition," which produces the Lucas component expression, the formula of which is given by

$$G_{i,j}(c,b) = [(2i+bj)G_{2,b} + (bi-2j)G_{b,-2}]/(b^2+4), \tag{1.8c}$$

where the denominator accounts for the normalization factor.

Since line-sequences $G_{x,y}$ and $G_{y,-x}$ are complementary, by repeated application of the vector addition and the scalar multiplication rules, see [8], we obtain the general orthogonal decomposition formula:

$$G_{i,j}(c,b) = [(xi+yj)G_{x,y} + (yi-xj)G_{y,-x}]/(x^2+y^2). \tag{1.8d}$$

Putting $x=1$ and $y=-1$, and applying the rule for scalar multiplication by -1, we obtain (1.8a); if we put $x=2$ and $y=b$, we obtain (1.8c).

Similarly, for an arbitrary pair of line-sequences $G_{x,y}$ and $G_{z,w}$, we find

$$G_{i,j}(c,b) = [-(wi-zj)G_{x,y} + (yi-xj)G_{z,w}]/(yz-wx). \tag{1.8e}$$

This is the general decomposition formula. For convenience, we call $G_{x,y}$ and $G_{z,w}$ the pair of coordinate line-sequences, and their coefficients the respective components. Putting $z=y$ and $w=-x$, we get (1.8d); if we put $x=z=1$, $y=\alpha$, and $w=\beta$, we get (1.8b). Wang and Zhang [11] adopted a very special pair of coordinates based on their conjugation property: $G_{0,1}$ and $G_{2,b}$. Putting $x=0$, $y=1$, $z=2$, and $w=b$, we obtain

$$G_{i,j}(c,b) = [-(bi-2j)G_{0,1} + iG_{2,b}]/2, \tag{1.8f}$$

which is equivalent to equation (2) in [11]. This decomposition scheme is particularly convenient to use in treating products of terms because of the conjugation property.

9. Translational Representation. By applying the translational relation (1.2a) to (1.8a), we obtain the translational representation of a general line-sequence in terms of the first basis,

$$G_{i,j}(c,b) = (iI + jc^{-1}T)G_{1,0}(c,b), \tag{1.9a}$$

where I denotes the identity translation; or, in terms of the elements, in the second basis:

$$g_n[i,j] = cig_{n-1}[0,1] + jg_n[0,1]. \tag{1.9b}$$

10. Binet's Product. Consistent with the multiplication of the corresponding terms in two line-sequences to obtain their product, we present the following definition.

Definition 2: The product of two line-sequences is defined as the product of the two respective Binet formulas, and shall be referred to as "Binet's Product." Also, for convenience, exponentiation notation is adopted when it applies.

Note that, except for some special cases, Binet's product does not, in general, constitute a line-sequence governed by (1.0a). This question will be discussed in a later paper.

In the line-sequential format, we then have

$$G_{1,0}G_{0,1} = (-(\beta G_{1,\alpha}^2 + \alpha G_{1,\beta}^2) + b(-c)^n)/(b^2+4c) \tag{1.10a}$$

or, in terms of the elements,

$$g_n[1,0]g_n[0,1] = (cg_{2n-1}[2,b] + b(-c)^n)/(b^2+4c). \tag{1.10c}$$

The conjugation of $G_{0,1}$ and $G_{2,b}$ (ref. (4.7) and (4.8) in [8]) is then given by

$$G_{0,1}G_{2,b} = (G_{1,\alpha}^2 - G_{1,\beta}^2)/(\alpha-\beta) \tag{1.10c}$$

or, in terms of the elements,

$$g_n[0,1]g_n[2,b] = g_{2n}[0,1]. \tag{1.10d}$$

This is the general conjugation formula relating the Fibonacci and the Lucas elements. For $c=b=1$, it reduces to the basic conjugation relation, $f_n l_n = f_{2n}$.

The Binet product of $G_{1,0}$ and $G_{b,-2}$ is somewhat more complex, that is,

$$G_{1,0}G_{b,-2} = \{[\beta(\beta b+2)G_{1,\alpha}^2 + \alpha(\alpha b+2)G_{1,\beta}^2] + 2b(c-1)G_{1,\alpha}G_{1,\beta}\}/(\alpha-\beta)^2 \tag{1.10e}$$

or, in terms of the elements,

$$g_n[1,0]g_n[b,-2] = c\{bcg_{2n-2}[2,b] - 2g_{2n-1}[2,b] + 2b(-c)^{n-1}(c-1)]/(b^2+4c). \tag{1.10f}$$

When $c=b=1$, it reduces to the more easily recognizable relation,

$$f_n[1,0]l_n[1,-2] = (l_{2n-2}[2,1] - 2l_{2n-1}[2,1])/5, \tag{1.10g}$$

which is a set of even terms in the *negative* Fibonacci line-sequence,

$$F_{0,-1}: \ldots, \mathbf{8}, -5, \mathbf{3}, -2, \mathbf{1}, -1, [\mathbf{0}, -1], -\mathbf{1}, -2, -\mathbf{3}, -5, -\mathbf{8}, \ldots. \tag{1.10h}$$

11. Summation. From the recurrence relation, it is easy to show that the general consecutive terms summation formula is given by

$$(b+c-1)\sum_{i=k}^{k+n} u_i = cu_{k+n} + u_{k+n+1} + (b-1)u_k - u_{k+1}, \tag{1.11a}$$

where $i \geq k$, $n \geq 0$; $i, k, n \in \mathbf{Z}$. We stress that this formula, like (4.3u) in [6], is translationally covariant. In the harmonic case, $b=c=1$, it reduces to the latter. In the case of Jacobsthal numbers, it reduces to (2.7) and (2.8) in [3], respectively; in the case of Jacobsthal polynomials, it reduces to (3.7) and (3.8) in [2], respectively, and so forth.

A word about the convention. In (4.3u) in [6], the translational degree of freedom is implicit in that the zero$^{\text{th}}$ element u_0 may be assigned to any element in the line-sequence. In the current case, however, we want to assign the zero$^{\text{th}}$ element in the formula to the zero$^{\text{th}}$ element of the line-sequence in question; for example, in the Pell line-sequence, we want to assign $u_0 = p_0$, so the translational degree of freedom lies explicitly in the value of the parameter k chosen.

It is easy to show that the following two equations hold:

$$(c-1)\sum_{i=k}^{k+n} u_{2i} + b\sum_{i=k}^{k+n} u_{2i-1} = c(u_{2(k+n)} - u_{2(k-1)}), \tag{1.11b}$$

$$(c-1)\sum_{i=k}^{k+n} u_{2i+1} + b\sum_{i=k}^{k+n} u_{2i} = c(u_{2(k+n)+1} - u_{2k-1}). \tag{1.11c}$$

In the harmonic case, (1.11b) reduces to the *odd* terms summation formula (4.4u), and (1.11c) reduces to the *even* terms summation formula (4.5u) in [6], respectively.

Combining (1.11b) and (1.11c), we obtain the general even terms summation formula,

$$\sum_{i=k}^{k+n} u_{2i} = [(c-1)(u_{2(k+n)+2} - u_{2k}) - bc(u_{2(k+n)+1} - u_{2k-1})]/[(c-1)^2 - b^2], \tag{1.11d}$$

and the general odd terms summation formula,

$$\sum_{i=k}^{k+n} u_{2i+1} = [c^2(u_{2(k+n)+1} - u_{2k-1}) - (u_{2(k+n)+3} - u_{2k+1})]/[(c-1)^2 - b^2]. \tag{1.11e}$$

12. Translational Operators. By the dual relation of Section 4 in [6], corresponding to formulas (1.11a) through (1.11e), we have the following set of covariant equations of the translational operators:

$$(b+c-1)\sum_{i=k}^{k+n} T_i = cT_{k+n} + T_{k+n+1} + (b-1)T_k - T_{k+1}, \tag{1.12a}$$

$$(c-1)\sum_{i=k}^{k+n} T_{2i} + b\sum_{i=k}^{k+n} T_{2i-1} = c(T_{2(k+n)} - T_{2(k-1)}), \tag{1.12b}$$

$$(c-1)\sum_{i=k}^{k+n} T_{2i+1} + b\sum_{i=k}^{k+n} T_{2i} = c(T_{2(k+n)+1} - T_{2k-1}), \tag{1.12c}$$

$$\sum_{i=k}^{k+n} T_{2i} = [(c-1)(T_{2(k+n)+2} - T_{2k}) - bc(T_{2(k+n)+1} - T_{2k-1})]/[(c-1)^2 - b^2], \tag{1.12d}$$

$$\sum_{i=k}^{k+n} T_{2i+1} = [c^2(T_{2(k+n)+1} - T_{2k-1}) - (T_{2(k+n)+3} - T_{2k+1})]/[(c-1)^2 - b^2]. \tag{1.12e}$$

13. Simson's Formula. The general Simson formula is found to be

$$g_{n+1}[i, j]g_{n-1}[i, j] - (g_n[i, j])^2 = (-c)^{n-1}(bij + ci^2 - j^2). \tag{1.13}$$

In particular, for the general Fibonacci and the general Lucas pairs,

$$g_{n+1}[1, 0]g_{n-1}[1, 0] - (g_n[1, 0])^2 = -(-c)^n, \tag{1.13a}$$

$$g_{n+1}[0, 1]g_{n-1}[0, 1] - (g_n[0, 1])^2 = -(-c)^{n-1}, \tag{1.13b}$$

$$g_{n+1}[2, b]g_{n-1}[2, b]-(g_n[2, b])^2 = (-c)^{n-1}(b^2+4c), \tag{1.13c}$$

$$g_{n+1}[b, -2]g_{n-1}[b, -2]-(g_n[b, -2])^2 = -(-c)^{n-1}(2b^2-b^2c+4). \tag{1.13d}$$

In the case of Jacobsthal numbers, (1.13b) and (1.13c) above reduce to (2.5) and (2.6) in [3], respectively. In the case of Jacobsthal polynomials, they reduce to (3.5) and (3.6) in [2], respectively, and so forth. From (1.13), it is clear that the significance of Simson's formula lies in its independence of the index n, apart from a sign correction.

2. THE GENERAL LUCAS PAIR

The general Lucas line-sequence $G_{2,b}$ is particularly interesting, mainly owing to its being conjugate to the second basis line-sequence $G_{0,1}$. In addition to the aforementioned properties, a few more basic properties are given below.

The basis component expression of $G_{2,b}$, according to (1.8a), is given by

$$G_{2,b} = 2G_{1,0} + bG_{0,1} \tag{2.1a}$$

or, in terms of the elements,

$$g_n[2, b] = 2g_n[1, 0] + bg_n[0, 1]. \tag{2.1b}$$

Substitution of the translational relation (1.2a) into (2.1a) produces the translational representation of $G_{2,b}$ in terms of the first basis,

$$G_{2,b} = (2I + bc^{-1}T)G_{1,0}, \tag{2.1c}$$

which can also be obtained from (1.9a) by putting $i=2$ and $j=b$.

The basis component expression of $G_{b,-2}$ is given by

$$G_{b,-2} = bG_{1,0} - 2G_{0,1} \tag{2.2a}$$

or, in terms of the elements,

$$g_n[b, -2] = bg_n[1, 0] - 2g_n[0, 1]. \tag{2.2b}$$

The translational representation in terms of the first basis is then given by

$$G_{b,-2} = (bI - 2c^{-1}T)G_{1,0}, \tag{2.2c}$$

which can again be obtained from (1.9a) by putting $i=b$ and $j=-2$.

Binet's formula for $G_{2,b}$, according to (1.8b), is

$$G_{2,b}(c, b) = G_{1,\alpha} + G_{1,\beta}, \tag{2.3a}$$

and Binet's formula for its complement is

$$G_{b,-2}(c, b) = [-(\beta b+2)G_{1,\alpha} + (\alpha b+2)G_{1,\beta}]/(\alpha-\beta). \tag{2.3b}$$

Substituting the geometrical line-sequences (1.5a) and (1.5b) into (2.3a) and noting that $\alpha\beta = -c$, we obtain the parity relation of the elements in $G_{2,b}$, that is,

$$g_{-n}[2, b] = (-c)^{-n}g_n[2, b].$$

Again in the nomenclature of Shannon and Horadam [9], the parity relation (2.4) reduces to (1.9) in [1] for $c=-1$ in the case of Morgan-Voyce even Lucas polynomials.

Applying the cross relation (1.4a) and the parity relation (1.3b) to the component expression (1.13b), using the parity relation (2.4), we obtain

$$g_n[2, b] = 2g_{n+1}[0, 1] - bg_n[0, 1], \tag{2.5}$$

which is the general version of the basis representation of the Lucas elements (for $c = b = 1$): $l_n = 2f_{n+1} - f_n$.

Similarly, from the component expression (2.2b), we obtain

$$g_n[b, -2] = bcg_{n-1}[0, 1] - 2g_n[0, 1], \tag{2.6a}$$

which is the basis representation of the complementary Lucas elements in terms of the second basis. Note that if we choose to express the elements in terms of the first basis, using the translational relation (1.2b), we would obtain

$$g_n[b, -2] = -2c^{-1}g_{n+1}[1, 0] + bg_n[1, 0], \tag{2.6b}$$

which is more symmetrical with (2.5).

3. THE PELL POLYNOMIAL LINE-SEQUENCES

We now apply the formulas obtained in the previous sections to the Pell polynomials and, for the sake of checking, we also calculate the results independently, that is, without using those formulas. The results are found to agree in each and every case. The order of development follows largely that of the previous sections with some minor variations.

The Pell polynomials line-sequence is defined by $b = 2x$, $c = 1$. The basic pair is given by

$$P_{1,0}(1, 2x): \ldots, -4x(1+2x^2), (1+4x^2), -2x, [1, 0], 1, 2x, (1+4x^2), \ldots, \tag{3.1a}$$

$$P_{0,1}(1, 2x): \ldots, -4x(1+2x^2), 1+4x^2, -2x, 1, [0, 1], 2x, (1+4x^2), \ldots, \tag{3.1b}$$

where the first one is referred to as the complementary P-Fibonacci line-sequence, or the $P_{1,0}$ line-sequence for short; the second is referred to as the P-Fibonacci line-sequence, or the $P_{0,1}$ line-sequence for short.

Obviously, they are translationally related, in agreement with (1.2a), that is,

$$TP_{1,0} = P_{0,1}. \tag{3.2a}$$

In terms of the elements, this becomes

$$p_{n+1}[1, 0] = p_n[0, 1]. \tag{3.2b}$$

The parity relation of the elements in $P_{1,0}$ is given by (1.3a),

$$p_{-n}[1, 0] = (-1)^n p_{n+2}[1, 0], \tag{3.3a}$$

and the parity relation of the elements in $P_{0,1}$ is given by (1.3b),

$$p_{-n}[0, 1] = (-1)^{n+1} p_n[0, 1]. \tag{3.3b}$$

Or, by applying (3.2b) to (3.3b), we also obtain (3.3a).

Combining the translational relations with the parity ones, we obtain the following set of cross relations among the elements of the two basis polynomial line-sequences, in agreement with relations (1.4a) through (1.4d):

$$p_{-n}[1,0]=(-1)^n p_{n+1}[0,1] \quad (3.4a)$$

$$p_{-n}[1,0]=p_{-(n+1)}[0,1]; \quad (3.4b)$$

or

$$p_{-n}[0,1]=(-1)^{n+1}p_{n+1}[1,0], \quad (3.4c)$$

$$p_{-n}[0,1]=p_{-(n-1)}[1,0]. \quad (3.4d)$$

From (1.5a) and (1.5b), the pair of geometrical line-sequences relating to $P_{1,0}$ is given by

$$P_{1,\alpha}(1,2x):\ldots,\alpha^{-2},\alpha^{-1},[1,\alpha],\alpha^2,\alpha^3,\ldots, \quad (3.5a)$$

$$P_{1,\beta}(1,2x):\ldots,\beta^{-2},\beta^{-1},[1,\beta],\beta^2,\beta^3,\ldots, \quad (3.5b)$$

respectively, where α and β are the roots of the generating equation

$$q^2-2xq-1=0. \quad (3.5c)$$

By formulas (1.6a) and (1.6b), Binet's formula for $P_{1,0}$ is

$$P_{1,0}=(-\beta P_{1},\alpha+\alpha P_{1},\beta)/(\alpha-\beta), \quad (3.6a)$$

and for the $P_{0,1}$ is

$$P_{0,1}=(P_{1,\alpha}-P_{1,\beta})/(\alpha-\beta). \quad (3.6b)$$

From (1.7a) and (1.7b), the P-Lucas line-sequence is given by

$$P_{2,2x}(1,2x):-2x(3+4x^2),2(1+2x^2),-2x,[2,2x],2(1+2x^2),2x(3+4x^2),\ldots. \quad (3.7a)$$

Its complement is then

$$P_{2x,-2}(1,2x):\ldots,2x(3+4x^2),-2(1+2x^2),[2x,-2],-2x,-2(1+2x^2),\ldots. \quad (3.7b)$$

These two line-sequences are clearly orthogonal, with a normalization factor $[2(1+x^2)^{1/2}]^{-1}$.

The basis component expression for an arbitrary Pell polynomial line-sequence, according to (1.8a), is given by

$$P_{i,j}(1,2x)=iP_{1,0}(1,2x)+jP_{0,1}(1,2x), \quad (3.8)$$

so we have, for the P-Lucas pair:

$$P_{2,2x}(1,2x)=2P_{1,0}(1,2x)+2xP_{0,1}(1,2x), \quad (3.8a)$$

$$P_{2x,-2}(1,2x)=2xP_{1,0}(1,2x)-2P_{0,1}(1,2x). \quad (3.8b)$$

It can be easily shown that the general formula of Binet decomposition, in the simpler applicable form for the Pell polynomial line-sequences, is given by

$$P_{i,j}(1,2x)=[-i(\beta P_{1,\alpha}-\alpha P_{1,\beta})+j(P_{1,\alpha}-P_{1,\beta})]/(\alpha-\beta). \quad (3.9)$$

Thus, we have

$$P_{2,2x}(1,2x)=2[-(\beta P_{1,\alpha}-\alpha P_{1,\beta})+x(P_{1,\alpha}-P_{1,\beta})]/(\alpha-\beta). \quad (3.9a)$$

$$P_{2x,-2}(1,2x)=2[-x(\beta P_{1,\alpha}-\alpha P_{1,\beta})+(P_{1,\alpha}-P_{1,\beta})]/(\alpha-\beta). \quad (3.9b)$$

The formula for the Lucas decomposition of an arbitrary Pell polynomial line-sequence, according to (1.8c), is given by

$$P_{i,j}(1, 2x) = [(i + xj)P_{2,2x} + (xi - j)P_{2x,-2}] / 2(1 + x^2), \tag{3.10}$$

so the component formulas of the basis pair with respect to the Lucas bases are

$$P_{1,0}(1, 2x) = [P_{2,2x} + xP_{2x,-2}] / 2(1 + x^2), \tag{3.10a}$$

$$P_{0,1}(1, 2x) = [xP_{2,2x} - P_{2x,-2}] / 2(1 + x^2). \tag{3.10b}$$

The conjugation of $P_{0,1}$ and $P_{2,b}$, by (1.10c), produces

$$P_{0,1}P_{2,2x} = (P_{1,\alpha}^2 - P_{1,\beta}^2) / (\alpha - \beta). \tag{3.11a}$$

In terms of the elements, this becomes

$$p_n[0, 1]p_n[2, 2x] = p_{2n}[0, 1], \tag{3.11b}$$

which is the P-version of the conjugation relation $f_n l_n = f_{2n}$.

The Binet product of $P_{1,0}$ and $P_{2x,-2}$, by (1.10e), is found to be

$$P_{1,0}P_{2x,-2} = 2[\beta(\beta x + 1)P_{1,\alpha}^2 + \alpha(\alpha x + 1)P_{1,\beta}^2] / (\alpha - \beta)^2. \tag{3.12a}$$

In terms of the elements, with $\alpha\beta = -1$, this becomes

$$p_n[1, 0]p_n[2x, -2] = (xp_{2n-2}[2, 2x] - p_{2n-1}[2, 2x]) / 2(1 + x^2). \tag{3.12b}$$

From Binet's formula (3.9a), we obtain the following parity relation between the elements of the P-Lucas line-sequence (3.7a),

$$p_{-n}[2, 2x] = (-1)^n p_n[2, 2x], \tag{3.13}$$

which apparently holds true.

The component expression of the P-Lucas line-sequence is given by

$$P_{2,2x} = 2P_{1,0} + 2xP_{0,1}. \tag{3.14a}$$

In terms of the elements, this becomes

$$p_n[2, 2x] = 2p_n[1, 0] + 2xp_n[0, 1]. \tag{3.14b}$$

Applying (3.3b) and (3.4a) and using the parity relation (3.13), we obtain

$$p_n[2, 2x] = 2p_{n+1}[0, 1] - 2xp_n[0, 1], \tag{3.14c}$$

which is the P-version of the relation $l_n = 2f_{n+1} - f_n$.

Substituting the translation relation (3.2a) to the component expression (3.14a), we obtain the translational representation of the P-Lucas line-sequence,

$$P_{2,2x} = 2(I + xT)P_{1,0}. \tag{3.15}$$

The component expression of the complementary P-Lucas line-sequence is

$$P_{2x,-2} = 2xP_{1,0} - 2P_{0,1}. \tag{3.16a}$$

In terms of the elements, this becomes

$$p_n[2x, -2] = 2xp_n[1, 0] - 2p_n[0, 1]. \tag{3.16b}$$

Applying parity relation (3.3b) and cross relation (3.4a), we obtain

$$p_n[2x,-2]=-2p_n[0,1]+2xp_{n-1}[0,1], \tag{3.16c}$$

which is the complement of the relation (3.14c) in terms of the second basis. Its equivalence in terms of the first basis is obtained by applying the translational relation (3.2b),

$$p_n[2x,-2]=-2p_{n+1}[1,0]+2xp_n[1,0], \tag{3.16d}$$

which is more symmetrical with (3.14c).

Substituting the translation relation (3.2a) into the component expression (3.16a), we obtain the translational representation of the complementary P-Lucas line-sequence,

$$P_{2x,-2}=2(xI-T)P_{1,0}. \tag{3.17}$$

The following summation formulas can be verified easily:

$$\sum_{i=k}^{k+n} p_i=[p_{k+n}+p_{k+n+1}+(2x-1)p_k-p_{k+1}]/2x, \tag{3.18a}$$

$$\sum_{i=k}^{k+n} p_{2i-1}=(p_{2(k+n)}-p_{2(k-1)})/2x, \tag{3.18b}$$

$$\sum_{i=k}^{k+n} p_{2i}=(p_{2(k+n)+1}-p_{2k-1})/2x. \tag{3.18c}$$

Formulas (1.11b) and (1.11c) in the general case reduce to (3.18b) and (3.18c), respectively.

The dual relation then gives the corresponding operators equations of translation:

$$\sum_{i=k}^{k+n} T_i=[T_{k+n}+T_{k+n+1}+(2x-1)T_k-T_{k+1}]/2x, \tag{3.19a}$$

$$\sum_{i=k}^{k+n} T_{2i-1}=(T_{2(k+n)}-T_{2(k-1)})/2x, \tag{3.19b}$$

$$\sum_{i=k}^{k+n} T_{2i}=(T_{2(k+n)+1}-T_{2k-1})/2x. \tag{3.19c}$$

For example, let $k=-3$ and $n=5$, then the left-hand side (l.h.s.) of (3.19a) gives

$$\left(\sum_{i=k}^{k+n} T_i\right)p_1[0,1]=3+4x^2,$$

and its right-hand side (r.h.s.) gives $[T_2+T_3+(2x-1)T_{-3}-T_{-2}]p_1[0,1]/2x=3+4x^2$; hence, l.h.s. = r.h.s.

Simson's formulas for the Pell polynomial line-sequence are found to be:

$$p_{n+1}[1,0]p_{n-1}[1,0]-(p_n[1,0])^2=(-1)^{n+1}, \tag{3.20a}$$

$$p_{n+1}[0,1]p_{n-1}[0,1]-(p_n[0,1])^2=(-1)^n, \tag{3.20b}$$

$$p_{n+1}[2,2x]p_{n-1}[2,2x]-(p_n[2,2x])^2=(-1)^{n-1}(4)(1+x^2), \tag{3.20c}$$

$$p_{n+1}[2x,-2]p_{n-1}[2x,-2]-(p_n[2x,-2])^2=(-1)^n(4)(1+x^2). \tag{3.20d}$$

For example, let $n=-1$ in (3.20d), then l.h.s. = r.h.s. = $-4(1+x^2)$.

Remark: A number of specific problems in this work need to be addressed. For example, as of this writing, we have not yet found the parity relation of the elements in $G_{b,-2}$, as compared to those in $G_{2,b}$, see (2.4). Also, as far as this author is aware of, the relation (3.12b) does not seem to relate to any known line-sequential relation, in contradistinguishing to relation (3.11b), which relates to the well-known conjugation relation $f_n l_n = f_{2n}$. It is also interesting to see, as is pointed out by the referee, that viewing from the bigger picture, so to say, how this piece of 2D work relates to the work in the 3D case, as, for example, in the context of [10].

ACKNOWLEDGMENT

The author wishes to express his gratitude to the anonymous referee for valuable comments on this report and also for suggesting reference [10].

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AMS Classification Numbers: 11B39, 15A03

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