SUMS OF n-th POWERS OF ROOTS OF A GIVEN QUADRATIC EQUATION

N.A. Draim, Ventura, Calif., and Marjorie Bicknell
Wilcox High School, Santa Clara, Calif.

The quadratic equation whose roots are the sum or difference of the n-th powers of the roots of a given quadratic equation is of interest to the algorithmatist both for the number patterns to be seen and for the implied connection between quadratic equations and the factorability and primeness of numbers. In this paper, an elementary approach which points out relationships which might otherwise be missed is used to derive general expressions for the subject equation and to show how Lucas and Fibonacci numbers arise as special cases.

Sums of like powers of roots of a given equation have been studied in detail. For a few examples, in this Quarterly S. L. Basin [1] has used a development of Waring's formula [2] for sums of like powers of roots to write a generating function for Lucas numbers and to write the kth Lucas number as a sum of binomial coefficients. R. G. Buschman [3] has used a linear combination of like powers of roots of a quadratic to study the difference equation $u_{n+1} = au_n + bu_{n-1}$, arriving at a result which expresses u_n in terms of a and b. Paul F. Byrd [4], [5] has studied the coefficients in expansions of analytic functions in polynomials associated with Fibonacci numbers. A number of the results in the foregoing references appear as special cases of those of the present paper.

A. THE SUMS OF THE n-th POWERS

Given the primitive equation $f(x) = x^2 - px - q = 0$, with roots r_1 and r_2, the first problem considered here is to write the equation

$$f(x)^n = x^2 - (r_1^n + r_2^n)x - (r_1 r_2)^n = 0 \quad ,$$

with roots r_1^n and r_2^n.

Solving the primitive equation for r_1 and r_2 by the quadratic formula,

$$r_1 = (p + \sqrt{p^2 + 4q})/2 , \qquad r_2 = (p - \sqrt{p^2 + 4q})/2 \quad .$$

Since r_1 and r_2 satisfy the equation $(x - r_1)(x - r_2) = 0$,

$$x^2 - (r_1 + r_2)x + r_1 r_2 = x^2 - px - q = 0$$

identically, so $r_1 + r_2 = p$ and $r_1 r_2 = -q$. Next, write the equation $f(x)^2$, having roots r_1^2 and r_2^2. Squaring roots,

$$r_1^2 = \left[(p + \sqrt{p^2 + 4q})/2\right]^2 = (2p^2 + 4q + 2p\sqrt{p^2 + 4q})/4 \; ,$$

$$r_2^2 = \left[(p - \sqrt{p^2 + 4q})/2\right]^2 = (2p^2 + 4q - 2p\sqrt{p^2 + 4q})/4 \; .$$

Adding, $r_1^2 + r_2^2 = p^2 + 2q$. The desired equation is then

$$f(x)^2 = x^2 - (p^2 + 2q)x + q^2 = 0 \; .$$

Similarly, the coefficient of x for the equation $f(x)^3 = 0$, by multiplying and adding, is $r_1^3 + r_2^3 = p^3 + 3pq$. Continuing in this manner, the coefficients for x for the equations whose roots are higher powers of r_1 and r_2 can be tabulated as follows:

The sequence for $r_1^n + r_2^n$ ends when, for a given n, the last term in the sequence becomes equal to zero. The penultimate term assumes the form $2q^{n/2}$ when n is even and $npq^{(n-1)/2}$ when n is odd.

Writing the numerical coefficients from Table 1 in the form of an array of m columns and n rows and letting $C(m, n)$ denote the coefficient in the mth column and nth row leads to Table 2. It is obvious that the iterative operation in Table 2 to generate additional numerical coefficients is

$$C(m, n) = C(m-1, n-2) + C(m, n-1) \quad ,$$

which suggests a linear combination of binomial coefficients. By direct expansion,

$$\frac{n(n-m)(n-m-1)\,\ldots\,(n-2m+3)}{(m-1)!} = 2\binom{n+1-m}{m-1} - \binom{n-m}{m-1} \quad .$$

TABLE 1: $r_1^n + r_2^n$

$$r_1 + r_2 = p$$
$$r_1^2 + r_2^2 = p^2 + 2q$$
$$r_1^3 + r_2^3 = p^3 + 3pq$$
$$r_1^4 + r_2^4 = p^4 + 4p^2q + 2q^2$$
$$r_1^5 + r_2^5 = p^5 + 5p^3q + 5pq^2$$
$$r_1^6 + r_2^6 = p^6 + 6p^4q + 9p^2q^2 + 2q^3$$
$$r_1^7 + r_2^7 = p^7 + 7p^5q + 14p^3q^2 + 7pq^3$$
$$r_1^8 + r_2^8 = p^8 + 8p^6q + 20p^4q^2 + 16p^2q^3 + 2q^4$$
$$\cdots \qquad \cdots$$
$$r_1^n + r_2^n = p^n + np^{n-2}q + \frac{n(n-3)}{2!}p^{n-4}q^2 + \frac{n(n-4)(n-5)}{3!}p^{n-6}q^3 + \cdots + \frac{n(n-m)(n-m-1)\ldots(n-2m+3)}{(m-1)!}p^{n-2m+2}q^{m-1}$$

TABLE 2: $C(m, n)$

n \ m	1	2	3	4	5	$\Sigma C(m, n)$
1	1	0				1
2	1	2				3
3	1	3	0			4
4	1	4	2			7
5	1	5	5	0		11
6	1	6	9	2		18
7	1	7	14	7	0	29
8	1	8	20	16	2	47

From the above tables,

$$(A.1)\qquad r_1^n + r_2^n = \sum_{i=0}^{[n/2]} \left[2\binom{n-i}{i} - \binom{n-i-1}{i}\right] p^{n-2i}q^i \quad ,$$

where $[x]$ is the greatest integer less than or equal to x, and $\binom{m}{n}$ is the binomial coefficient

$$\frac{m!}{(m-n)!\ n!}$$

Proof: By algebra,

$$\begin{aligned} r_1^{k+1} + r_2^{k+1} &= (r_1^{k+1} + r_2^{k+1} + r_1 r_2^k + r_1^k r_2) - (r_1 r_2^k + r_1^k r_2) \\ &= (r_1^k + r_2^k)(r_1 + r_2) - r_1 r_2 (r_1^{k-1} + r_2^{k-1}) \\ &= p(r_1^k + r_2^k) + q(r_1^{k-1} + r_2^{k-1}) \quad . \end{aligned}$$

From Table 1, (A.1) holds for $n = 1, 2, \ldots, 8$. To prove (A.1) by mathematical induction, assume that (A.1) holds when $n = k$ and $n = k-1$. Using the result just given and the inductive hypothesis,

$$\begin{aligned} r_1^{k+1} + r_2^{k+1} = p &\sum_{i=0}^{[k/2]} \left[2\binom{k-i}{i} - \binom{k-i-1}{i}\right] p^{k-2i}q^i \\ &+ q \sum_{i=0}^{[(k-1)/2]} \left[2\binom{k-1-i}{i} - \binom{k-i-2}{i}\right] p^{k-1-2i}q^i \quad . \end{aligned}$$

Multiplying as indicated and using the index substitution $i-1$ for i in the second series yields

$$\begin{aligned} r_1^{k+1} + r_2^{k+1} = &\sum_{i=0}^{[k/2]} \left[2\binom{k-i}{i} - \binom{k-i-1}{i}\right] p^{k+1-2i}q^i \\ &+ \sum_{i=0}^{[(k+1)/2]} \left[2\binom{k-i}{i-1} - \binom{k-i-1}{i-1}\right] p^{k+1-2i}q^i \quad . \end{aligned}$$

Since the recursion formula for binomial coefficients is

$$\binom{m}{n} + \binom{m}{n+1} = \binom{m+1}{n+1} \quad ,$$

combining the series above will yield

$$\left[2\binom{k+1-i}{i} - \binom{k-i}{i}\right]$$

for the coefficient of $p^{k+1-2i}q^i$. Considering the series for k even and for k odd leads to

$$r_1^{k+1} + r_2^{k+1} = \sum_{i=0}^{[(k+1)/2]} \left[2\binom{k+1-i}{i} - \binom{k-i}{i}\right] p^{k+1-2i}q^i \quad ,$$

so that (A.1) follows by mathematical induction. An expression equivalent to (A.1) was given by Basin in [1].

Using formula (A.1), we can write the desired equation

$$f(x)^n = x^2 - (r_1^n + r_2^n)x + (-1)^n q^n = 0 \quad .$$

Example: Given the equation $x^2 - 5x + 6 = 0$, write the equation whose roots are the fourth powers of the roots of the given equation, without solving the given equation. In the given equation, $p = 5$, $q = 6$. From Table 1 or formula (A.1),

$$r_1^4 + r_2^4 = (5)^4 + 4(5)^2(-6) + 2(-6)^2 = 97 \quad ,$$

so $f(x)^4 = x^2 - 97x + 6^4 = 0$. As a check, by factoring $x^2 - 5x + 6 = 0$, $r_1 = 2$ and $r_2 = 3$, giving us $f(x)^4 = (x - 2^4)(x - 3^4) = x^2 - 97x + 6^4$.

Returning to Table 2, the summation of the coefficients by rows yields the series 1, 3, 4, 7, 11, 18, 29, 47, ..., the successive Lucas numbers defined by $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$. The general equation for $r_1^n + r_2^n$ reduces to the numerical values of $C(m, n)$ of Table 2 for $p = q = 1$ in the primitive equation $x^2 - px - q = 0$, which becomes $x^2 - x - 1 = 0$ with roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. This yields an expression for the nth Lucas number, since $L_n = \alpha^n + \beta^n$;

$$\text{(A. 2)} \qquad L_n = 1 + n + n(N-3)/2! + n(n-4)(n-5)/3! + n(n-5)(n-6)(n-7)/4!$$

$$= \sum_{i=0}^{[n/2]} \left[2\binom{n-i}{i} - \binom{n-i-1}{i} \right] .$$

For example, $L_6 = 1 + 6 + 6(3)/2 + 6(2)(1)/6 = 18$.

Also, the equation whose roots are the nth roots of $x^2 - x - 1 = 0$ is $x^2 - L_n x + (-1)^n = 0$. (Essentially, equation (A. 2), as well as (A. 3), (B. 2), (B. 3), appears in [3] by Buschman.)

An alternate expression for L_n is obtained by expressing $r_1^n + r_2^n$ in terms of p and Δ, where $\Delta = \sqrt{p^2 + rq}$. Now,

$$r_1^n = [(p + \Delta)/2]^n = 2^{-n}(p^n + np^{n-1}\Delta + \frac{n(n-1)}{2!} p^{n-2}\Delta^2 + \frac{n(n-1)(n-2)}{3!} p^{n-3}\Delta^3 + \ldots)$$

$$r_2^n = [(p - \Delta)/2]^h = 2^{-n}(p^n - np^{n-1}\Delta + \frac{n(n-1)}{2!} p^{n-2}\Delta^2 + \frac{n(n-1)(n-2)}{3!} p^{n-3}\Delta^3 + \ldots)$$

which, for $p = q = 1$, $\Delta = \sqrt{5}$, implies, on adding,

$$\text{(A. 3)} \qquad L_n = 2^{1-n}(1 - 5n(n-1)/2! + 5^2 n(n-1)(n-2)(n-3)/4! + \ldots)$$

$$= \frac{1}{2^{n-1}} \sum_{i=0}^{[n/2]} 5^i \binom{n}{2i} .$$

B. THE DIFFERENCES OF THE nth POWERS

Now let us turn to the equation whose roots are r_1^n and $-r_2^n$, where r_1 and r_2 are the roots of the primitive equation $x^2 - px - q = 0$. Since

$$r_1 - r_2 = (p + \sqrt{p^2 + rq})/2 - (p - \sqrt{p^2 + 4q})/2 = \sqrt{p^2 + 4q} = \Delta \quad ,$$

the equation whose roots are r_1 and $-r_2$ is $x^2 - \sqrt{p^2 + rq}\, x - r_1 r_2 = 0$, and $f(x)_\Delta = x^2 - \Delta x + q = 0$. Similarly,

$$r_1^2 - r_2^2 = [(p + \sqrt{p^2 + 4q})/2]^2 - [(p - \sqrt{p^2 + 4q})/2]^2 = p\Delta \quad .$$

Continuing in this manner, and assembling results, we have

TABLE 3: $r_1^n - r_2^n$

n	$r_1^n - r_2^n$
1	Δ
2	$p\Delta$
3	$(p^2 + q)\Delta$
4	$(p^3 + 2pq)\Delta$
5	$(p^4 + 3p^2q + q^2)\Delta$
6	$(p^5 + 4p^3q + 3pq^2)\Delta$
7	$(p^6 + 5p^4q + 6p^2q^2 + q^3)\Delta$
...	...
n	$(p^{n-1} + (n-2)p^{n-3}q + \frac{(n-3)(n-4)}{2!}p^{n-5}q^2 + \ldots + \frac{(n-m-1)(n-m-2)\ldots(n-2m)}{m!}p^{n-2m-1}q^m)\Delta$

The sequence for $r_1^n - r_2^n$ as given in Table 3 ends, for a given n, on arriving at term zero.

Writing the numerical coefficients from Table 3 in the form of an array of m columns and n rows, and letting $C_\Delta(m, n)$ denote the coefficient in the mth column and nth row,

TABLE 4: $C_\Delta(m, n)$

n \ m	1	2	3	4	5	6	$\Sigma C_\Delta(m, n)$
1	1	0					1
2	1	0					1
3	1	1	0				2
4	1	2	0				3
5	1	3	1	0			5
6	1	4	3	0			8
7	1	5	6	1	0		13
8	1	6	10	4	0		21
9	1	7	15	10	1	0	34

It is obvious that the iterative process to generate additional numerical coefficients rests on the relation

$$C_\Delta(m, n) = C_\Delta(m-1,\ n-2) + C_\Delta(m,\ n-1) \quad ,$$

the same as for Table 2. The coefficients generated are also the coefficients appearing in the Fibonacci polynomials defined by

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x), \quad f_1(x) = 1, \quad f_2(x) = x \quad .$$

(For further references to Fibonacci polynomials, see Byrd, [4], p. 17.)

Tables 3 and 4 lead to

$$\text{(B.1)} \qquad r_1^n - r_2^n = \Delta \sum_{j=0}^{[n/2]} \binom{n-j-1}{j} p^{n-j-1} q^j \quad ,$$

which is proved similarly to (A.1)

The equation whose roots are r_1^n and $-r_2^n$, where r_1 and r_2 are the roots of $x^2 - px - q = 0$, is $f(x)_\Delta^n = x^2 - (r_1^n - r_2^n)x + q^n = 0$. We use (B.1) to solve the following: Given the equation $x^2 - 10x + 21 = 0$ with unsolved roots r_1 and r_2, write the quadratic equation whose roots are r_1^3 and $-r_2^3$. From the given equation, $p = 10$, $q = -21$, and $\Delta = \sqrt{10^2 - 4(21)} = 4$. Using (B.1) or Table 3,

$$r_1^3 - r_2^3 = (p^2 + q) \quad = (10^2 - 21)4 = 316 \quad .$$

Hence $f(x)_\Delta^3 = x^2 - 316x + (-21)^3 = 0$. As a check, the roots of the problem equation are $r_1 = 7$, $r_2 = 3$, so $f(x)_\Delta^3 = (x - 7^3)(x + 3^3) = x^2 - 316x - 21^3 = 0$.

The summation of terms in the successive rows of Table 4 is seen to yield the sequence of Fibonacci numbers, defined by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. This result can be demonstrated by substituting $p = q = 1$ in the general expression (B.1) and dividing by $\Delta = \sqrt{5}$, for the left hand member becomes the Binet form, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$. In general,

(B.2) $$F_n = 1 + (n-2) + (n-3)(n-4)/2! + (n-4)(n-5)(n-6)/3! + \dots$$
$$= \sum_{j=0}^{[n/2]} \binom{n-j-1}{j} ,$$

the sum of terms in rising diagonals of Pascal's triangle. Also, the quadratic equation whose roots are α^n and $-\beta^n$, nth powers of the roots of the quadratic equation $x^2 - x - 1 = 0$ with roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$, is $x^2 - \sqrt{5}\ F_n x - 1 = 0$.

Expressing $r_1^n - r_2^n$ in terms of only p and Δ and assembling terms as we did for $r_1^n + r_2^n$ in (A.3), we arrive at an alternate expression for the nth Fibonacci number,

(B.3) $$F_n = 2^{1-n}(n + 5n(n-1)(n-2)/3! + 5^2 n(n-1)(n-2)(n-3)(n-4)/5! + \dots)$$
$$= 2^{1-n} \sum_{i=0}^{[(n-1)/2]} 5^i \binom{n}{2i+1} .$$

REFERENCES

1. S. L. Basin, "A Note on Waring's Formula for Sums of Like Powers of Roots", Fibonacci Quarterly 2:2, April, 1964, pp. 119-122.
2. L. E. Dickson, First Course in the Theory of Equations, John Wiley, 1922.
3. R. G. Buschman, "Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations", Fibonacci Quarterly 1:4, Dec., 1963, pp. 1-8.
4. Paul F. Byrd, "Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers", Fibonacci Quarterly, 1:1, Feb. 1963, pp. 16-29.
5. Paul F. Byrd, "Expansion of Analytic Functions in Terms Involving Lucas Numbers or Similar Number Sequences", Fibonacci Quarterly 3:2, April, 1965, pp. 101-114.

xxxxxxxxxxxxxx