FIBONACCI AND EUCLID

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For many centuries number addicts have had much fun with the numbers generated by the Pythagorean identity

$$(2mn)^2 + (m^2-n^2)^2 = (m^2+n^2)^2$$

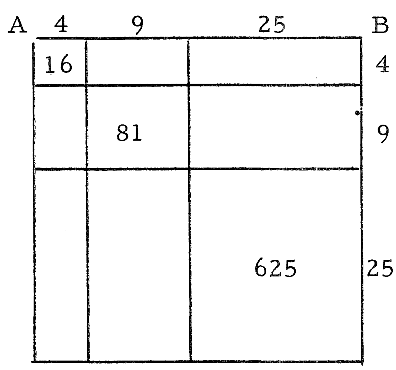
Candido's identity viz.

$$[m^2+n^2+(m+n)^2]^2 = 2\,[m^4+n^4+(m+n)^4]$$

may have its moments too, especially for Fibonacci fans. It is easily verified by expanding both sides and obviously holds for all integral values of m and n. Therefore it holds when m and n are consecutive Fibonacci or Lucas numbers. It has its application to geometry as illustrated by the following problem.

> "Divide a given straight line into three segments so that the square on the whole line is twice the sum of the squares on the three segments."

Solution: Construct a square on line AB. Let AB (and the remaining sides) be divided into three segments which are measured by the second powers of any three consecutive Fibonacci numbers, e.g., 2^2, 3^2, 5^2 or 4, 9, 25.



Then the cross lines joining the points of division divide the square into nine rectangles — three are squares and six are non-squares. Thus we see that the great square is twice the sum of the three smaller squares, i.e.,

$$38^2 = 2\,[16 + 81 + 625] = 1444.$$

Furthermore the six non-squares are equal in pairs and therefore represent the six faces of a cuboid. The area and diagonal of the large square are equal respectively to twice the area and diagonal of the cuboid. This identity can be applied to the areas of circles and spheres on the line and its segments or to any similar polygons drawn on the same.

Problemists may find many applications to geometry, e.g., "From the corners of an equilateral triangle cut off similar triangles so that half the area remains."

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Now if

$$\frac{b^2}{8} - \frac{b}{4} = 1 \quad (b = 1 \pm 3) \quad ,$$

equation (6) has the solutions

$$L(s) = \begin{cases} \cos s \\ \cosh s \end{cases} .$$

Therefore

$$(7) \qquad F(s) = 2 \begin{Bmatrix} \cos s \\ \cosh s \end{Bmatrix} + \frac{b}{2} = 2 \begin{Bmatrix} \cos s \\ \cosh s \end{Bmatrix} + \frac{1 \pm 3}{2} .$$

Equation (5) now gives

$$(8) \qquad u_k = 2 \cosh A2^k + \frac{1 \pm 3}{2}$$

for the hyperbolic cosine alternative in equation (6). Now if A is chosen to be Arccosh $\frac{n_0}{2}$ where n_0 is an odd integer, it is easily shown that the first term of the right number of equation (8) is always an odd integer and hence that u_k is not divisible by 2 with the choice of the positive sign. A similar result holds for n_0 even with the negative sign. Therefore by the theorem, the sequence (b) is relatively prime. The cosine alternative in equation (7) leads to a bounded sequence of integers and therefore is not very interesting.

REFERENCE

1. Hardy, G. H. and Wright, E. M., "An Introduction to the Theory of Numbers", Oxford, 1960.

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