A NOTE ON A THEOREM OF JACOBI

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It has been shown that the sequence, L_n, or $1, 3, 4, 7, 11, 18, 29, 47, \cdots$ is defined by

$$(1) \qquad L_n = u^n + (-u)^{-n} \qquad (u = (1 + \sqrt{5})/2) \quad ,$$

where L_n is the n^{th} Lucas number.

Theorem 1. If $L_{2^{n-1}} = c_n$ $(n = 2, 3, 4, \cdots)$, p is a prime $4m + 3$, $M_p = 2^p - 1$ and

$$F(x) = \prod_{n=2}^{\infty}\left(1 - x^{2^{n-1}}\right)\left(1 + 3x^{2^{n-1}-1} + x^{2^n-2}\right) = 1 + \sum_{n=2}^{\infty} c_n x^{2^{n-1}} \quad ;$$

Then, M_p is a prime if and only if $c_p \equiv 0 \pmod{M_p}$.

Proof. Using the famous identity of Jacobi from Hardy and Wright [1, p. 282],

$$(2) \quad \prod_{n=1}^{\infty}((1 - x^{2n})(1 + x^{2n-1}z)(1 + x^{2n-1}z^{-1})) = 1 + \sum_{n=1}^{\infty} x^{n^2}(z^n + z^{-n})$$

we put $z + z^{-1} = 3$ $(z = (3 + \sqrt{5})/2)$, and combining z with u in (1) we have $u^2 = z$, so that $L_{2n} = z^n + z^{-n}$, which leads to

$$(3) \quad \prod_{n=1}^{\infty}(1 - x^{2n})(1 + 3x^{2n-1} + x^{4n-2}) = 1 + \sum_{n=1}^{\infty} L_{2n}x^{n2} \quad .$$

Next, in Theorem 1 we put $L_{2^{n-1}} = c_n$ and replace n with 2^{n-2}, where it is evident the resulting equation is identical to $F(x)$.

We complete the proof of Theorem 1 with the following theorem of Lucas appearing in [2, p. 397]:

... If $4m + 3$ is prime, $P = 2^{4m+3} - 1$ is prime if the first term of the series $3, 7, 47, \cdots$, defined by $r_{n+1} = r_n^2 - 2$, which is divisible by P is of rank $4m + 2$; but P is composite if no one of the first $4m + 2$ terms is divisible by P... .

Corollary. If $[x]$ denotes the greatest integer contained in x and $n!/(n-r)!r! = \binom{n}{r}$, then

$$z^n + z^{-n} = n \sum_{r=0}^{[n/2]} (-1)^r (n-r)^{-1} \binom{n-r}{r} b^{n-2r} \quad .$$

The proof of the Corollary is obtained by elementary means if we put

$$z^n + z^{-n} = ((b + \sqrt{b^2 - 4})/2)^n + ((b + \sqrt{b^2 - 4})/2)^{-n}$$

and then add the right side of the equation.

In conclusion, although there are many special cases to the Corollary, the one obtained by setting $b = 0$ may be worth mentioning.

Let

$p(n)$ = the number of unrestricted partitions of an integer n,

$p_m(n)$ = the number of partitions of n into parts not exceeding m, where $p(0) = p_m(0) = 1$.

We then have the following:

Theorem 2. If

$$F_m(x) = \prod_{n=1}^{m} (1 - x^n)^{-1} = 1 + \sum_{n=1}^{\infty} p_m(n) x^n$$

and

$$\prod_{n=1}^{\infty} (1 - x^n)^{-1} = 1 + \sum_{n=1}^{\infty} p(n) x^n \quad ,$$

then

$$p(2u) \equiv p_2(u - 2) + p_4(u - 8) + \cdots + p_{2r}(u - 2r^2) + \cdots \pmod 2$$

and

$$p(2u + 1) \equiv p_1(u) + p_3(u - 4) + \cdots + p_{2r+1}(u - 2r^2 - 2r) + \cdots \pmod 2.$$

<u>Proof</u>. Putting $z + z^{-1} = 0$, then $z = i$ $(i^2 = -1)$, so that $z^{2n} + z^{-2n} = 2(-1)^n$. Then applying these results, together with our replacing x with x^2 in Eqs. (2) and (3), leads to

$$\prod_{n=1}^{\infty}(1 - x^n)(1 + x^{2n-1}) = 1 + 2\sum_{r=1}^{\infty}(-1)^r x^{2r^2}$$

and it is evident that

$$(4) \qquad \prod_{n=1}^{\infty}(1 + x^{2n-1}) \equiv \sum_{n=0}^{\infty} p(n)x^n \pmod 2 .$$

According to Hardy and Wright [1, p. 281], Euler proved that

$$\prod_{n=1}^{\infty}(1 + x^{2n-1}) = 1 + \sum_{r=1}^{\infty} x^{r^2} F_r(x^2) \quad ,$$

and combining this result with $F_m(x)$ in Theorem 2 and with Eq. (4), we complete the proof of Theorem 2.

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REFERENCES

1. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, fourth ed., Oxford University Press, London, 1959.

2. L. E. Dickson, Theory of Numbers, Publication No. 256, Vol. I.

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