BINOMIAL SUMS OF FIBONACCI POWERS

JOHN WESSNER, Melbourne, Florida

At the impetus of Professor Hoggatt, a general solution was obtained for summations of the form

$$\sum_{k=0}^{n} \binom{n}{k} (\pm 1)^k F_k^b \quad ,$$

where $b = 2m$. Some suggestions will be made for attacking the problem for odd b, and it is hoped that a complete solution will be forthcoming.

Let us first consider the case where $b = 4p$. If $\alpha = 1/2(1 + \sqrt{5})$, $\beta = 1/2\ (1 - \sqrt{5})$, $F_k = (\alpha^k + \beta^k)/(\alpha - \beta)$, and $L_k = \alpha^k + \beta^k$, then

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} F_k^{4p} &= \sum_{k=0}^{n} \binom{n}{k} 5^{-2p} (\alpha^k - \beta^k)^{4p} \\
&= \sum_{k=0}^{n} \binom{n}{k} 5^{-2p} \sum_{t=0}^{4p} \binom{4p}{t} (-1)^t (\alpha^{4p-t} \beta^t)^k \\
&= 5^{-2p} \sum_{k=0}^{n} \binom{n}{k} \left\{ \sum_{j=0}^{2p-1} \binom{4p}{j} (-1)^{j(k+1)} \left[(\alpha^{4p-2j})^k + (\beta^{4p-2j})^k \right] \right. \\
&\qquad \left. + \binom{4p}{2p} (-1)^{2p(k+1)} \right\} \\
&= 5^{-2p} \sum_{j=0}^{2p-1} \binom{4p}{j} (-1)^j \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^{jk} L_{(4p-2j)k} \right\} + 5^{-2p} \binom{4p}{2p} 2^n ,
\end{aligned}$$

where we have made use of the fact that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \ .$$

Hence, we have reduced the problem to one involving Lucas numbers and no powers. We must digress to obtain the required Lucas formulas.

Consider

$$L(n,1,q) = \sum_{k=0}^{n} \binom{n}{k} L_{qk} = \sum_{k=0}^{n} \binom{n}{k} (\alpha^{qk} + \beta^{qk})$$
$$= (1 + \alpha^q)^n + (1 + \beta^q)^n .$$

If $q = 2g$, then

$$L(n,1,2g) = \left\{(\alpha\beta)^g(-1)^g + \alpha^{2g}\right\}^n + (\alpha\beta)^g(-1)^g + \beta^{2g}\}^n$$
$$= \alpha^{gn}\left\{\alpha^g + (-1)^g\beta^g\right\}^n + \beta^{gn}\left\{(-1)^g\alpha^g + \beta^g\right\}^n .$$

The manipulation of this depends upon the parity of g. Let $g = 2r$, or $q = 4r$:

$$L(n,1,4r) = (\alpha^{2rn} + \beta^{2rn})(\alpha^{2r} + \beta^{2r})^n = L_{2rn}L_{2r}^n .$$

If, instead, $q = 4r + 2$, then for odd values of n we obtain

$$L(n,1,4r+2) = (\alpha^{(2r+1)n} + (-1)^n\beta^{(2r+1)n})(\alpha^{2r+1} - \beta^{2r+1})^n$$
$$= 5^{\frac{1}{2}(n+1)}F_{(2r+1)n}F_{2r+1}^n .$$

For even values of n,

$$L(n,1,4r+2) = 5^{\frac{1}{2}n}L_{(2r+1)n}F_{2r+1}^n .$$

By the same methods we obtain

$$L(n,-1,4r+2) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{(4r+2)k} = (-1)^n L_{(2r+1)n}L_{2r+1}^n ,$$

and

$$L(n, -1, 4r) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{4rk} = 5^{\frac{1}{2}n} L_{2rn} F_{2r}^n$$

for even n, and

$$L(n, -1, 4r) = -5^{\frac{1}{2}(n+1)} F_{2rn} F_{2r}^n$$

if n is odd.

Now, let us return to the original problem.

$$\sum_{k=0}^{n} \binom{n}{k} F_k^{4p} = 5^{-2p} \binom{4p}{2p} 2^n$$

$$+ 5^{-2p} \sum_{j=0}^{2p-1} \binom{4p}{j} (-1)^j \sum_{k=0}^{n} \binom{n}{k} (-1)^{jk} L_{(4p-2j)k}$$

$$= 5^{-2p} \left\{ 2^n \binom{4p}{2p} + \sum_{i=0}^{p-1} \binom{4p}{2i} L(n,1,4[p-i]) - \sum_{i=1}^{p} \binom{4p}{2i-1} L(n,-1,4[p-i]+2) \right\}$$

$$= 5^{-2p} \left\{ 2^n \binom{4p}{2p} + \sum_{i=0}^{p-1} \binom{4p}{2i} L_{2(p-i)n} L_{2(p-i)}^n \right.$$

$$\left. - (-1)^n \sum_{i=1}^{n} \binom{4p}{2i-1} L_{2(p-i)n+n} L_{2(p-i)+1}^n \right\} .$$

Similarly, if $b = 4p + 2$,

$$\sum_{k=0}^{n} \binom{n}{k} F_k^{4p+2} = 5^{-(2p+1)} \sum_{k=0}^{n} \binom{n}{k} \left\{ \sum_{j=0}^{2p} \binom{4p+2}{j} (-1)^{j(k+1)} [(\alpha^{4p-2j+2})^k \right.$$

$$\left. + (\beta^{4p-2j+2})^k] + \binom{4p+2}{2p+1} (-1)^{(k+1)(2p+1)} \right\}$$

$$= 5^{-(2p+1)} \sum_{j=0}^{2p} \binom{4p+2}{j} (-1)^j \sum_{k=0}^{n} \binom{n}{k} (-1)^{jk} L_{(4p-2j+2)k}$$

$$+ 5^{-(2p+1)} \binom{4p+2}{2p+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1}$$

$$= 5^{-(2p+1)} \left\{ \sum_{i=0}^{p} \binom{4p+2}{2i} L(n,1,4[p-i]+2) - \sum_{i=0}^{p-1} \binom{4p+2}{2i+1} L(n,-1,4[p-i]) \right\}.$$

For summations involving alternating signs the same method of analysis gives similar results.

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k^{4p} = 5^{-2p} \left\{ \sum_{i=0}^{p-1} \binom{4p}{2i} L(n,-1,4[p-i]) - \sum_{i=1}^{p} \binom{4p}{2i-1} L(n,1,4[p-i]+2) \right\},$$

and

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k^{4p+2} = 5^{-(2p+1)} \left\{ \sum_{i=0}^{p} \binom{4p+2}{2i} L(n,-1,4[p-i]+2) - \sum_{i=0}^{p-1} \binom{4p+2}{2i+1} L(n,1,4[p-i]) - \binom{4p+2}{2p+1} 2^n \right\}.$$

Finally, a word regarding summations for odd powers of F_k. For powers $b \geqq 5$, the problem still reduces to a binomial sum involving Lucas numbers whose subscripts are in an arithmetic progression, but the expressions $(1 + \alpha^q)$ and $(1 + \beta^q)$ cannot be reduced in a manner similar to that used for even q. Surely they are reducible, and it is hoped that the expression obtained above may be extended to odd powers.

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