COMPOSITION OF RECURSIVE FORMULAE

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The purpose of this article is to consider some properties of composite functions, such as F_{F_n}, L_{F_n}, etc. For ease of notation, F(n) and L(n) will represent the usual Fibonacci and Lucas sequences, respectively. The following notations will also be adopted:

$$g(n) \equiv L\{F(n)\}$$
$$h(n) \equiv F\{F(n)\}$$
$$f(n) \equiv F\{L(n)\}$$
$$k(n) \equiv L\{L(n)\}$$

Part I. Recursive Relations for g(n), h(n), f(n), k(n).

Although hybrid relations for the above were sought, only partial success was achieved.

(1) $$5h(n)h(n+1) = g(n+2) = (-1)^{F(n)}g(n-1)$$

Proof. In this and subsequent parts, considerable use was made of the well known identities,

(a) $$\sqrt{5}F(n) = \alpha^n - \beta^n$$

(b) $$L(n) = \alpha^n + \beta^n$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2} .$$

In (a) replace n by F(n) and obtain

$$\sqrt{5}h(n) = \alpha F(n) - \beta F(n)$$
$$\sqrt{5}h(n+1) = \alpha F(n+1) - \beta F(n+1)$$

Multiplying these and observing

$$F(n+2) = F(n) + F(n+1)$$
$$F(n+1) = F(n) + F(n-1)$$
$$\alpha\beta = -1$$

the result follows.

(2) $$\frac{g(n)g(n+1) - g(n+2)}{g(n-1)} = (-1)^{F(n)}$$

Proof. In (b) replace n by F(n) and then as above the result follows.

(3) $$g(n)g(n+1) + 5h(n)h(n+1) = 2g(n+2)$$

Proof. Combine (1) and (2).

(4) $$5f(n)f(n+1) = k(n+2) - (-1)^{L(n)}k(n-1)$$

Proof. In (a) replace n by L(n) and as above the result follows.

(5) $$\frac{k(n)k(n+1) - k(n+2)}{k(n-1)} = (-1)^{L(n)}$$

Proof. In (b) replace n by L(n) and as in (2) the result follows upon rearrangement.

(6) $$k(n)k(n+1) + 5f(n)f(n+1) = 2k(n+2)$$

Proof. Combine (4) and (5).

(7) $$g(n+1)g(n-1) = k(n) + (-1)^{F(n-1)}g(n) .$$

Proof. In (b) replace n by F(n) and observe that

$$L(n) = F(n+1) + F(n-1).$$

The result then follows.

(8) $$5h(n-1)h(n+1) = k(n) = (-1)^{F(n-1)}g(n).$$

Proof. In (a) replace n by F(n - 1) and F(n + 1) and multiply.

(9) $$g(n+1)g(n-1) + 5h(n-1)h(n+1) = 2k(n)$$

Proof. Combine (7) and (8).

(10) $$g(n+1)g(n-1) - 5h(n-1)h(n+1) = 2(-1)^{F(n-1)}g(n).$$

Proof. As above, combining (7) and (8) yields the result.

By combining the above relationships, many others may be obtained. Note that (2) and (5) are hybrid relations, whereas the others are mongrel. It would be of interest to obtain hybrid relations for h(n) and f(n). In the light of the above results one can scarcely help setting up the correspondences

$$h(n) \longleftrightarrow f(n)$$

$$g(n) \longleftrightarrow k(n).$$

Part II. Explicit Relations for $g(n)$, $h(n)$, $f(n)$, $k(n)$.

Using (a) and (b) one immediately obtains the rather clumsy formulae

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\begin{aligned}
g(n) &= \alpha^{\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)} + \beta^{\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)} \\
h(n) &= \frac{1}{\sqrt{5}}\left\{\alpha^{\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)} - \beta^{\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)}\right\} \\
f(n) &= \frac{1}{\sqrt{5}}\left\{\alpha^{(\alpha^n+\beta^n)} - \beta^{(\alpha^n+\beta^n)}\right\} \\
k(n) &= \alpha^{(\alpha^n+\beta^n)} + \beta^{(\alpha^n+\beta^n)}
\end{aligned}
$$

Part III. Generating Functions for $g(n)$, $h(n)$, $f(n)$, $k(n)$.

The original goal of obtaining generating functions of the type

$$F(x) = \sum_0^\infty h(k)x^k ,$$

was not achieved.

The methods used by Riordan [1] did not appear to offer much help in this direction. However, ugly mixed types could be obtained as the following example illustrates:

$$\frac{x \ln \alpha}{1 - x^2} = \sum_0^\infty \ln\left\{\frac{\sqrt{5}h(i) + g(i)}{2}\right\} x^i$$

Proof.

$$\alpha^{F(n)} = \frac{\sqrt{5}h(n) + g(n)}{2}$$

Also, since

$$\frac{x}{1 - x - x^2} = \sum_0^\infty F_i x^i ,$$

the result follows.

Part IV. Upper Bounds for $h(n)$, $g(n)$, $f(n)$, $k(n)$.

Part II yields explicit values for $h(n)$, etc., but the relations are awkward to work with and more workable bounds were sought.

(1) $$h(n) \le 2^{2^{n-2}-2} \quad \text{for} \quad n \ge 3$$

Proof. $h(n)$, $g(n)$, $f(n)$, $k(n)$ are increasing functions of n, and it is well known that

$$F(n) \le 2^{n-2} \quad \text{for} \quad n \ge 3.$$

Hence

$$h(n) = F\{F(n)\} \le F(2^{n-2}) \le 2^{2^{n-2}-2}$$

(2) $$f(n) \le 2^{5(2)^{n-3}-2} \quad \text{for} \quad n \ge 3$$

<u>Proof.</u>

$$L(n) = F(n-1) + F(n+1) \le 2^{n-3} + 2^{n-1} = 5(2)^{n-3}$$

The result follows as above.

(3) $$k(n) \le 5(2)^{5(2)^{n-3}-3} \quad \text{for} \quad n \ge 3$$

<u>Proof.</u> The result follows as above.

(4) $$g(n) \le 5(2)^{2^{n-2}-3} \quad \text{for} \quad n \ge 3$$

<u>Proof.</u> As above.

The above inequalities may be replaced by strict inequalities for $n > 3$, since

$$F(n) < 2^{n-2} \quad \text{for} \quad n > 3$$

Proposal for Future Investigations

To reiterate, the two most interesting avenues for future work are the development of generating functions for $h(n)$, $g(n)$, $f(n)$, $k(n)$, and hybrid recursive relations for $h(n)$ and $f(n)$.

REFERENCE

1. John Riordan, "Generating Functions for Powers of Fibonacci Numbers," <u>Duke Math. J.</u>, 29 (1962), pp. 5-12.

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