SOME BINOMIAL CONVOLUTION FORMULAS

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0. INTRODUCTION

For a natural number ν and two sequences $\{A(k), B(k)\}_k$ of binomial coefficients, the following convolutions of Vandermonde type,

$$C(m,n,\nu) := \sum_k A(m+k\nu)B(n-k\nu),$$

will be investigated in this paper. When $\nu = 2$, 3, 4, the convolutions will be nominated duplicate, triplicate, and quadruplicate, respectively. Thanks to the explicit solutions of the corresponding algebraic equations, we will establish the generating functions of binomial coefficients with running indices multiplicated accordingly. Then the formal power series method will be used to demonstrate several binomial convolution identities.

When $\nu = 1$, we reproduce a pair of binomial identities and the related generating function relations, from which our argument will be developed. In this respect, there are two general convolution formulas due to Hagen and Rothe (cf. [9], §5.4),

$$\sum_{k=0}^{n} \frac{\alpha}{\alpha+k\beta}\binom{\alpha+k\beta}{k}\binom{\gamma-k\beta}{n-k}\frac{\gamma-n\beta}{\gamma-k\beta} = \frac{\alpha+\gamma-n\beta}{\alpha+\gamma}\binom{\alpha+\gamma}{n} \tag{0.1a}$$

and

$$\sum_{k=0}^{n} \frac{\alpha}{\alpha+k\beta}\binom{\alpha+k\beta}{k}\binom{\gamma-k\beta}{n-k} = \binom{\alpha+\gamma}{n}, \tag{0.1b}$$

which have been recovered by Gould [7] (see also [3], [6], and §4.5 in [10]) through manipulating the generating functions

$$\sum_{k=0}^{\infty} \frac{\alpha}{\alpha+k\beta}\binom{\alpha+k\beta}{k}\tau^k = \eta^\alpha \tag{0.2a}$$

and

$$\sum_{k=0}^{\infty} \frac{\alpha}{\alpha+k\beta}\binom{\alpha+k\beta}{k}\tau^k = \frac{\eta^{1+\alpha}}{\beta+\eta-\beta\eta}, \tag{0.2b}$$

where $\tau = (\eta-1)/\eta^\beta$. More binomial convolution formulas and the related hypergeometric identities may be found in [4] and [8].

For an indeterminate x and a complex sequence $\{T(k)\}_k$, the generating function is defined by the following formal power series:

$$f(x) = \sum_{k=0}^{\infty} T(k)x^k. \tag{0.3a}$$

Denote by $\omega_\nu = \exp(\frac{2\pi\sqrt{-1}}{\nu})$ the ν^{th} primitive root of unity. Then there exists a well-known formula to determine the generating function of the subsequence with running indices congruent to ι modulo ν,

$$\nu\sum_{k=0}^{\infty} T(k\nu+\iota)x^{k\nu+\iota} = \sum_{\kappa=0}^{\nu-1}\omega_\nu^{\kappa(\nu-\iota)} f(x\omega_\nu^\kappa), \tag{0.3b}$$

which will be used in this paper frequently without indication.

1. DUPLICATE CONVOLUTIONS

For $\beta = 1/2$, the functional equation between two variables τ and η in (0.2) becomes quadratic. The substitution of its solution $\eta(2\tau) \to U^2(\tau)$ leads the generating functions stated in (0.2a) and (0.2b) to the following lemma.

Lemma 1.1: For two indeterminates τ and U related by

$$2\tau = U - \frac{1}{U} \Leftrightarrow U = \tau + \sqrt{1+\tau^2}, \tag{1.1a}$$

we have functional equations

$$1 = U(\tau)\times U(-\tau), \tag{1.1b}$$

$$2\tau = U(\tau) - U(-\tau), \tag{1.1c}$$

$$2\sqrt{1+\tau^2} = U(\tau) + U(-\tau), \tag{1.1d}$$

$$1 + U^2(\tau) = \{U(\tau)+U(-\tau)\}U(\tau), \tag{1.1e}$$

$$1 + U^2(-\tau) = \{U(\tau)+U(-\tau)\}U(-\tau), \tag{1.1f}$$

and generating functions

$$\sum_{k=0}^{\infty} \frac{a}{a+k/2}\binom{a+k/2}{k}(2\tau)^k = U^{2a}(\tau) \tag{1.2a}$$

and

$$\sum_{k=0}^{\infty} \binom{a+k/2}{k}(2\tau)^k = \frac{2U^{1+2a}(\tau)}{U(\tau)+U(-\tau)}. \tag{1.2b}$$

Their combinations lead us immediately to the following proposition.

Proposition 1.2: With the complex function U defined in Lemma 1.1, we have generating functions on duplicated binomial coefficients:

$$U^{2a}(\tau) + U^{2a}(-\tau) = \sum_{k=0}^{\infty} \frac{2a}{a+k}\binom{a+k}{2k}(2\tau)^{2k}, \tag{1.3a}$$

$$U^{2a-1}(\tau) - U^{2a-1}(-\tau) = \sum_{k=0}^{\infty} \frac{2a-1}{a+k}\binom{a+k}{1+2k}(2\tau)^{2k+1}, \tag{1.3b}$$

$$\frac{U^{1+2a}(\tau) + U^{1+2a}(-\tau)}{U(\tau)+U(-\tau)} = \sum_{k=0}^{\infty} \binom{a+k}{2k}(2\tau)^{2k}, \tag{1.3c}$$

$$\frac{U^{2a}(\tau) - U^{2a}(-\tau)}{U(\tau)+U(-\tau)} = \sum_{k=0}^{\infty} \binom{a+k}{1+2k}(2\tau)^{2k+1}. \tag{1.3d}$$

Based on these relations, we are ready to establish binomial formulas on duplicate convolutions.

Theorem 1.3 (Duplicate convolution identities [5]):

$$\sum_k \frac{2a-m}{a+k}\binom{a+k}{m+2k}\binom{c-k}{n-2k} = \binom{a+c}{m+n} + (-1)^m \binom{c-a+m}{m+n}, \tag{1.4a}$$

$$\sum_k \frac{2a-m}{a+k}\binom{a+k}{m+2k}\binom{c-k}{n-2k}\frac{2c-n}{c-k} \tag{1.4b}$$

$$= \frac{2a+2c-m-n}{a+c}\binom{a+c}{m+n} + (-1)^m \frac{2c-2a+m-n}{c-a+m}\binom{c-a+m}{m+n}. \tag{1.4c}$$

Proof: By means of Lemma 1.1, manipulate generating functions

$$\{U^{2a}(\tau)+U^{2a}(-\tau)\} \times \frac{2U^{1+2c-n}(\tau)}{U(\tau)+U(-\tau)} = \frac{2U^{1+2a+2c-n}(\tau)}{U(\tau)+U(-\tau)} + \frac{2U^{1+2c-2a-n}(\tau)}{U(\tau)+U(-\tau)}$$

and

$$\{U^{2a-1}(\tau)-U^{2a-1}(-\tau)\} \times \frac{2U^{1+2c-n}(\tau)}{U(\tau)+U(-\tau)} = \frac{2U^{2a+2c-n}(\tau)}{U(\tau)+U(-\tau)} - \frac{2U^{2+2c-2a-n}(\tau)}{U(\tau)+U(-\tau)}.$$

According to Proposition 1.2, the coefficients of τ^n and τ^{1+n} in the formal power series expansions lead us, respectively, to the following convolution formulas,

$$\sum_k \frac{2a}{a+k}\binom{a+k}{2k}\binom{c-k}{n-2k} = \binom{a+c}{n} + \binom{c-a}{n} \tag{1.5a}$$

and

$$\sum_k \frac{2a-1}{a+k}\binom{a+k}{1+2k}\binom{c-k}{n-2k} = \binom{a+c}{1+n} - \binom{1+c-a}{1+n}, \tag{1.5b}$$

which have been discovered for the first time by Andrews-Burge (see [1], Eqs. 3.1-3.2), with the help of hypergeometric transformations in their work on plane partition enumerations and determinant evaluations.

Letting $\delta = 0,\ 1$ be the Kronecker delta, we can unify both formulas as a unilateral convolution identity,

$$\sum_k \frac{2a-\delta}{a+k}\binom{a+k}{\delta+2k}\binom{c-k}{n-2k} = \binom{a+c}{\delta+n} + (-1)^\delta \binom{\delta+c-a}{\delta+n},$$

which, in turn, is expressed under parameter replacements

$$\begin{aligned} &k \to k+p,\\ &a \to a-p,\\ &c \to c+p,\\ &n \to n+2p,\\ &\delta \to m-2p, \end{aligned}$$

as the first finite bilateral convolution formula stated in the theorem.

Again from Lemma 1.1 and Proposition 1.2, consider the generating functions

$$\{U^{2a}(\tau)+U^{2a}(-\tau)\} \times U^{2c-n}(\tau) = U^{2a+2c-n}(\tau) + U^{2c-2a-n}(\tau)$$

and

$$\{U^{2a-1}(\tau)-U^{2a-1}(-\tau)\} \times U^{2c-n}(\tau) = U^{2a+2c-n-1}(\tau) - U^{2c-2a-n+1}(\tau),$$

then the coefficients of τ^n and τ^{1+n} in the formal power series expansions result, respectively, in the following binomial convolution identities:

$$\sum_k \frac{2a}{a+k}\binom{a+k}{2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k} \tag{1.6a}$$

$$=\frac{2a+2c-n}{a+c}\binom{a+c}{n}+\frac{2c-2a-n}{c-a}\binom{c-a}{n}, \tag{1.6b}$$

$$\sum_k \frac{2a-1}{a+k}\binom{a+k}{1+2k}\frac{2c-n}{c-k}\binom{c-k}{n-2k} \tag{1.6c}$$

$$=\frac{2a+2c-n-1}{a+c}\binom{a+c}{1+n}-\frac{2c-2a-n+1}{1+c-a}\binom{1+c-a}{1+n}. \tag{1.6d}$$

Their bilateralization derived exactly in the same way as in the proof of the first formula (1.4a) leads us to the second one (1.4b-1.4c). This completes the proof of Theorem 1.3. □

As a by-product, we present a pair of convolution formulas of Jensen type. From Lemma 1.1, it is trivial to have the formal power series

$$\frac{1}{U(\tau)+U(-\tau)}=\frac{1}{2\{U(\tau)-\tau\}}=\frac{1}{2}\sum_{k=0}^{\infty}\frac{\tau^k}{U^{1+k}(\tau)}.$$

By means of Proposition 1.2, we can establish the following expansions,

$$\frac{U^{1+2a}(\tau)+U^{1+2a}(-\tau)}{U(\tau)+U(-\tau)}\times\frac{2U^{1+2c}(\tau)}{U(\tau)+U(-\tau)}=\sum_{k=0}^{\infty}\frac{\tau^k U^{1+2a+2c-k}(\tau)+\tau^k U^{1+2a-2c+k}(-\tau)}{U(\tau)+U(-\tau)}$$

and

$$\frac{U^{2a}(\tau)-U^{2a}(-\tau)}{U(\tau)+U(-\tau)}\times\frac{2U^{1+2c}(\tau)}{U(\tau)+U(-\tau)}=\sum_{k=0}^{\infty}\frac{\tau^k U^{2a+2c-k}(\tau)-\tau^k U^{2c-2a-k}(\tau)}{U(\tau)+U(-\tau)},$$

whose coefficients of τ^n and τ^{1+n} lead us, respectively, to the Jensen convolutions

$$\sum_k\binom{a+k}{2k}\binom{c-k}{n-2k}=\sum_\iota\frac{1}{2^{1+\iota}}\left\{\binom{a+c-\iota}{n-\iota}+\binom{a-c+n}{n-\iota}(-1)^{n-\iota}\right\} \tag{1.7a}$$

and

$$\sum_k\binom{a+k}{1+2k}\binom{c-k}{n-2k}=\sum_\iota\frac{1}{2^{1+\iota}}\left\{\binom{a+c-\iota}{1+n-\iota}-\binom{a-c-\iota}{1+n-\iota}\right\}. \tag{1.7b}$$

Further formulas of Jensen type and binomial identities related to Theorem 1.3 as well as their applications to determinant evaluations can be found in [2] and [5].

2. TRIPLICATE CONVOLUTIONS

When $\beta=1/3$, the functional equation between two variables τ and η in (0.2) is cubic. The substitution of its solution $\eta(3\tau)\to V^3(\tau)$ can be used to reformulate the generating functions stated in (0.2a)-(0.2b) as follows.

Lemma 2.1: Denote the cubic root of unity by $\varepsilon=\exp(2\pi i/3)$. For two indeterminates τ and V related by

$$3\tau = V^2 - \frac{1}{V} \Leftrightarrow V = \Lambda(\tau) + \frac{\tau}{\Lambda(\tau)}, \tag{2.1a}$$

where

$$\Lambda(\tau) = \sqrt[3]{\{1+\sqrt{1-4t^3}\}/2}, \tag{2.1b}$$

we have the functional equations

$$1 = V(\tau)\times V(\tau\omega)\times V(\tau\omega^2), \tag{2.2a}$$
$$0 = V(\tau) + \omega V(\tau\omega) + \omega^2 V(\tau\omega^2), \tag{2.2b}$$
$$-3\tau = V(\tau\omega)V(\tau\omega^2) + \omega V(\tau)V(\tau\omega) + \omega^2 V(\tau)V(\tau\omega^2), \tag{2.2c}$$

and generating functions

$$\sum_{k=0}^{\infty} \frac{a}{a+k/3}\binom{a+k/3}{k}(3\tau)^k = V^{3a}(\tau), \tag{2.3a}$$

$$\sum_{k=0}^{\infty} \binom{a+k/3}{k}(3\tau)^k = \frac{3V^{3+3a}(\tau)}{1+2V^3(\tau)}. \tag{2.3b}$$

Their combinations yield the following generating functions on triplicated binomial coefficients.

Proposition 2.2: With complex function V defined as in Lemma 2.1, we have generating function relations:

$$V^{3a}(\tau) + V^{3a}(\tau\omega) + V^{3a}(\tau\omega^2) = \sum_{k=0}^{\infty} \frac{3a}{a+k}\binom{a+k}{3k}(3\tau)^{3k}, \tag{2.4a}$$

$$V^{3a}(\tau) + \omega^2 V^{3a}(\tau\omega) + \omega V^{3a}(\tau\omega^2) = \sum_{k=0}^{\infty} \frac{3a}{\frac{1}{3}+a+k}\binom{\frac{1}{3}+a+k}{1+3k}(3\tau)^{1+3k}, \tag{2.4b}$$

$$V^{3a}(\tau) + \omega V^{3a}(\tau\omega) + \omega^2 V^{3a}(\tau\omega^2) = \sum_{k=0}^{\infty} \frac{3a}{\frac{2}{3}+a+k}\binom{\frac{2}{3}+a+k}{2+3k}(3\tau)^{2+3k}. \tag{2.4c}$$

Theorem 2.3 (Triplicate convolution identities): Given two natural numbers m and n, define

$$\theta(m,n) = \omega^{m+2n} + \omega^{2m+n} = \begin{cases} +2, & m \equiv n \pmod 3, \\ -1, & m \not\equiv n \pmod 3. \end{cases} \tag{2.5}$$

Then there holds a binomial identity

$$\sum_k \frac{3a}{\frac{m}{3}+a+k}\binom{\frac{m}{3}+a+k}{m+3k}\frac{a}{\frac{n}{3}+a-k}\binom{\frac{n}{3}+a-k}{n-3k} \tag{2.6a}$$

$$= \frac{2a}{\frac{m+n}{3}+2a}\binom{\frac{m+n}{3}+2a}{m+n} + \frac{-a}{\frac{m+n}{3}-a}\binom{\frac{m+n}{3}-a}{m+n}\theta(m,n) \tag{2.6b}$$

and its reversal

$$\sum_k \frac{3c}{\frac{2m}{3}+c+2k}\binom{\frac{2m}{3}+c+2k}{m+3k}\frac{c}{\frac{2n}{3}+c-2k}\binom{\frac{2n}{3}+c-2k}{n-3k} \tag{2.7a}$$

$$= \frac{2c}{\frac{2m+2n}{3}+2c}\binom{\frac{2m+2n}{3}+2c}{m+n} + \frac{-c}{\frac{2m+2n}{3}-c}\binom{\frac{2m+2n}{3}-c}{m+n}\theta(m,n). \tag{2.7b}$$

Proof: By means of Lemma 2.1, manipulate generating functions:

$$V^{3a}(\tau)\times\{V^{3a}(\tau)+V^{3a}(\tau\omega)+V^{3a}(\tau\omega^2)\}=V^{6a}(\tau)+V^{-3a}(\tau\omega)+V^{-3a}(\tau\omega^2),$$
$$V^{3a}(\tau)\times\{V^{3a}(\tau)+\omega^2V^{3a}(\tau\omega)+\omega V^{3a}(\tau\omega^2)\}=V^{6a}(\tau)+\omega V^{-3a}(\tau\omega)+\omega^2V^{-3a}(\tau\omega^2),$$
$$V^{3a}(\tau)\times\{V^{3a}(\tau)+\omega V^{3a}(\tau\omega)+\omega^2V^{3a}(\tau\omega^2)\}=V^{6a}(\tau)+\omega^2V^{-3a}(\tau\omega)+\omega V^{-3a}(\tau\omega^2).$$

According to Proposition 2.2, the coefficients of $\tau^{n+\nu}$, $\nu=0,\ 1,\ 2$, in the formal power series expansions lead us, respectively, to the following binomial convolutions,

$$\sum_k \frac{3a}{\frac{\nu}{3}+a+k}\binom{\frac{\nu}{3}+a+k}{\nu+3k}\frac{a}{\frac{n}{3}+a-k}\binom{\frac{n}{3}+a-k}{n-3k}$$
$$=\frac{2a}{\frac{n+\nu}{3}+2a}\binom{\frac{n+\nu}{3}+2a}{n+\nu}+\frac{-a}{\frac{n+\nu}{3}-a}\binom{\frac{n+\nu}{3}-a}{n+\nu}\theta(\nu,n),$$

which gives rise to the first finite bilateral convolution formula stated in the theorem under parameter replacements $k\to k+p$, $\nu\to m-3p$, and $n\to n+3p$. Rewriting every binomial coefficient in the first binomial identity through

$$\frac{\alpha}{\gamma}\binom{\gamma}{\ell}=(-1)^{\ell}\frac{-\alpha}{\ell-\gamma}\binom{\ell-\gamma}{\ell}$$

we immediately obtain the second one in the theorem. □

3. QUADRUPLICATE CONVOLUTIONS

For $\beta=1/4$, the functional equation between two variables τ and η in (0.2) becomes quartic. The substitution of its solution $\eta(4\tau)\to W^4(\tau)$ leads the generating functions stated in (0.2a)-(0.2b) to the following lemma.

Lemma 3.1: For two indeterminates τ and W related by

$$4\tau=W^3-\frac{1}{W}\Leftrightarrow W=\frac{\tau+\sqrt{\Omega^3(\tau)-\tau^2}}{\Omega},\tag{3.1a}$$

where

$$\Omega(\tau)=\sqrt{\phi^2(\tau)+\phi(\tau)\psi(\tau)+\psi^2(\tau)}\tag{3.1b}$$

and

$$\phi(\tau),\psi(\tau)=\sqrt[3]{\pm\tau^2+\sqrt{\tau^4+1/27}},\tag{3.1c}$$

we have the functional equations

$$\Omega(\tau)=\Omega(-\tau)=\Omega(i\tau)=\Omega(-i\tau),\tag{3.2a}$$
$$W(\tau)\times W(-\tau)\times W(i\tau)\times W(-i\tau)=1,\tag{3.2b}$$
$$\phi(\tau)\times\psi(\tau)=\tfrac{1}{3}\ \text{and}\ \phi^3(\tau)-\psi^3(\tau)=2\tau^2,\tag{3.2c}$$

and generating functions

$$\sum_{k=0}^{\infty}\frac{a}{a+k/4}\binom{a+k/4}{k}(4\tau)^k=W^{4a}(\tau),\tag{3.3a}$$

$$\sum_{k=0}^{\infty}\binom{a+k/4}{k}(4\tau)^k=\frac{4W^{4+4a}(\tau)}{1+3W^4(\tau)}. \tag{3.3b}$$

Their combinations bring about the following generating functions.

Proposition 3.2: With the complex function W as in Lemma 3.1, we have the following generating function relations:

$$\sum_{k=0}^{\infty}\frac{4c}{c+k}\binom{c+k}{4k}(4\tau)^{4k}=W^{4c}(\tau)+W^{4c}(-\tau) \tag{3.4a}$$

$$+W^{4c}(i\tau)+W^{4c}(-i\tau), \tag{3.4b}$$

$$\sum_{k=0}^{\infty}\frac{4c}{\frac{1}{4}+c+k}\binom{\frac{1}{4}+c+k}{1+4k}(4\tau)^{1+4k}=W^{4c}(\tau)-W^{4c}(-\tau) \tag{3.4c}$$

$$-iW^{4c}(i\tau)+iW^{4c}(-i\tau), \tag{3.4d}$$

$$\sum_{k=0}^{\infty}\frac{4c}{\frac{1}{2}+c+k}\binom{\frac{1}{2}+c+k}{2+4k}(4\tau)^{2+4k}=W^{4c}(\tau)+W^{4c}(-\tau) \tag{3.4e}$$

$$-W^{4c}(i\tau)-W^{4c}(-i\tau), \tag{3.4f}$$

$$\sum_{k=0}^{\infty}\frac{4c}{\frac{3}{4}+c+k}\binom{\frac{3}{4}+c+k}{3+4k}(4\tau)^{3+4k}=W^{4c}(\tau)-W^{4c}(-\tau) \tag{3.4g}$$

$$+iW^{4c}(i\tau)-iW^{4c}(-i\tau). \tag{3.4h}$$

Theorem 3.3 (Quadruplicate convolution identities): For two integers m and n, let

$$\varepsilon(m,n)=\begin{cases}+1, & m+n\not\equiv 0\ (\mathrm{mod}\ 4),\\ -1, & m+n\equiv 0\ (\mathrm{mod}\ 4).\end{cases} \tag{3.5}$$

Then, for $m\times n\not\equiv 1$ (mod 4), we have

$$\sum_k\frac{4c}{\frac{m}{4}+c+k}\binom{\frac{m}{4}+c-k}{m+4k}\frac{c}{\frac{n}{4}+c-k}\binom{\frac{n}{4}+c-k}{n-4k} \tag{3.6a}$$

$$+\sum_k\frac{-4c}{\frac{m}{4}-c+k}\binom{\frac{m}{4}-c+k}{m+4k}\frac{-c}{\frac{n}{4}-c-k}\binom{\frac{n}{4}-c-k}{n-4k}\varepsilon(m,n) \tag{3.6b}$$

$$=\frac{2c}{\frac{m+n}{4}+2c}\binom{\frac{m+n}{4}+2c}{m+n}+\frac{-2c}{\frac{m+n}{4}-2c}\binom{\frac{m+n}{4}-2c}{m+n}\varepsilon(m,n). \tag{3.6c}$$

Otherwise, for $m\times n\equiv 1$ (mod 4), there holds

$$\sum_k\frac{4c}{\frac{m}{4}+c+k}\binom{\frac{m}{4}+c-k}{m+4k}\frac{c}{\frac{n}{4}+c-k}\binom{\frac{n}{4}+c-k}{n-4k}\varepsilon(m,n) \tag{3.7a}$$

$$+\sum_k\frac{-4c}{\frac{2+m}{4}-c+k}\binom{\frac{m}{4}-c-k}{2+m+4k}\frac{-c}{\frac{n-2}{4}-c-k}\binom{\frac{n}{4}-c-k}{n-2-4k} \tag{3.7b}$$

$$=\frac{2c}{\frac{m+n}{4}+2c}\binom{\frac{m+n}{4}+2c}{m+n}\varepsilon(m,n)+\frac{-2c}{\frac{m+n}{4}-2c}\binom{\frac{m+n}{4}-2c}{m+n}. \tag{3.7c}$$

Proof: For $\nu = 0, 1, 2, 3$, define the binomial convolutions $\Diamond_\nu(n, c)$ by

$$\Diamond_\nu(n,c) = \sum_k \frac{4c}{\frac{\nu}{4}+c+k}\binom{\frac{\nu}{4}+c+k}{\nu+4k}\frac{c}{\frac{n}{4}+c-k}\binom{\frac{n}{4}+c-k}{n-4k}. \tag{3.8}$$

The proof of the theorem will be divided into four cases according to m (mod 4).

Case 1: $m \equiv 0 \pmod 4$. By means of Lemma 3.1 and (3.4a)-(3.4b), we may manipulate generating functions:

$$\begin{aligned}&-W^{8c}(\tau)+W^{4c}(\tau)\times\{W^{4c}(\tau)+W^{4c}(-\tau)+W^{4c}(i\tau)+W^{4c}(-i\tau)\}\\ &= W^{-4c}(i\tau)W^{-4c}(-i\tau)+W^{-4c}(-\tau)W^{-4c}(i\tau)+W^{-4c}(-\tau)W^{-4c}(-i\tau).\end{aligned}$$

The coefficients of τ^n in the formal power series expansions lead us to the following binomial convolutions,

$$\Delta_0(n,c) \stackrel{\text{def}}{=} \Diamond_0(n,c) - \frac{2c}{2c+\frac{n}{4}}\binom{2c+\frac{n}{4}}{n} \tag{3.9a}$$

$$= \sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{n-k}{4}-c}\binom{\frac{n-k}{4}-c}{n-k}\lambda_0(n,k), \tag{3.9b}$$

where $\lambda_0(n,k) = (-1)^n\{i^k + i^{3k} + i^{2k+3n}\}$, whose values are displayed in Table 1.

TABLE 1. Values of $\lambda_0(n, k)$

$k \backslash n$	0	1	2	3
0	3	$i-2$	1	$-2-i$
1	-1	$-i$	1	i
2	-1	$i+2$	-3	$2-i$
3	-1	$-i$	1	i

For $n \equiv 0 \pmod 2$, Table 1 suggests that we express (3.9b) as

$$\begin{aligned}\Delta_0(n,c) &= (-1)^{n/2}\sum_k \frac{-4c}{k-c}\binom{k-c}{4k}\frac{-c}{\frac{n}{4}-c-k}\binom{\frac{n}{4}-c-k}{n-4k}\\ &- (-1)^{n/2}\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{n-k}{4}-c}\binom{\frac{n-k}{4}-c}{n-k},\end{aligned}$$

which may be simplified, by means of (0.1a), to the following relation:

$$\Delta_0(n,c) = (-1)^{n/2}\Diamond_0(n,-c) - (-1)^{n/2}\frac{-2c}{\frac{n}{4}-2c}\binom{\frac{n}{4}-2c}{n} \tag{3.10a}$$

$$= (-1)^{n/2}\Delta_0(n,-c), \quad n \equiv 0 \pmod 2. \tag{3.10b}$$

While $n \equiv 1 \pmod 2$, it is easy to check from Table 1 that

$$\lambda_0(n,k)+\lambda_0(n,n-k) = -2(-1)^{\frac{k(n-k)}{2}}$$

Then the combination of (3.12b) and its reversal enables us to write

$$\Delta_0(n,c)=\sum_k \frac{c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{n-k}{4}-c}\binom{\frac{n-k}{4}-c}{n-k}(-1)^{\frac{k(n-k)}{2}}$$

$$=\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{n-k}{4}-c}\binom{\frac{n-k}{4}-c}{n-k}-4\sum_{k\equiv 0\,(\mathrm{mod}\,4)}\frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{n-k}{4}-c}\binom{\frac{n-k}{4}-c}{n-k}.$$

Applying (0.1a) to the penultimate sum, we get

$$\Delta_0(n,c)\ =-\Diamond_0(n,-c)+\frac{-2c}{\frac{n}{4}-2c}\binom{\frac{n}{4}-2c}{n} \tag{3.11a}$$

$$=-\Delta_0(n,-c),\quad n\equiv 1\ (\mathrm{mod}\,2). \tag{3.11b}$$

Both relations (3.10) and (3.11) may be stated as the single one $\Delta_0(n,c)+\Delta_0(n,-c)\varepsilon(0,n)=0$ which, in view of (3.9a), confirms the case $m\equiv 0$ (mod 4) of Theorem 3.3 with replacements $k\to k+p$ and $n\to n+4p$.

Case 2: $m\equiv 1$ (mod 4). In view of Lemma 3.1 and (3.4c)-(3.4d), we have the generating function relation:

$$-W^{8c}(\tau)+W^{4c}(\tau)\times\{W^{4c}(\tau)-W^{4c}(-\tau)-iW^{4c}(i\tau)+iW^{4c}(-i\tau)\}$$
$$=-W^{-4c}(i\tau)W^{-4c}(-i\tau)+iW^{-4c}(-\tau)W^{-4c}(i\tau)-iW^{-4c}(-\tau)W^{-4c}(-i\tau).$$

The coefficients of τ^{1+n} in their formal power series expansions leads us to the following binomial convolutions,

$$\Delta_1(n,c)\stackrel{\text{def}}{=}\Diamond_1(n,c)-\frac{2c}{2c+\frac{1+n}{4}}\binom{2c+\frac{1+n}{4}}{1+n} \tag{3.12a}$$

$$=\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{1+n-k}{4}-c}\binom{\frac{1+n-k}{4}-c}{1+n-k}\lambda_1(n,k), \tag{3.12b}$$

where $\lambda_1(n,k)=(-1)^n\{i^{1+k}+i^{1+n+2k}+i^{3+3k}\}$, whose values are displayed in Table 2.

TABLE 2. Values of $\lambda_1(n,k)$

$k \backslash n$	0	1	2	3
0	i	1	$-i$	-1
1	$-2-i$	1	$i-2$	3
2	i	1	$-i$	-1
3	$2-i$	-3	$i+2$	-1

For $n\equiv 1$ (mod 2), Table 2 suggests that we rewrite (3.12b) as

$$\Delta_1(n,c)=(-1)^{\frac{n-1}{2}}\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{1+n-k}{4}-c}\binom{\frac{1+n-k}{4}-c}{1+n-k}$$
$$-(-1)^{\frac{n-1}{2}}\sum_k \frac{-4c}{\frac{3}{4}-c+k}\binom{\frac{3}{4}-c+k}{3+4k}\frac{-c}{\frac{n-2}{4}-c-k}\binom{\frac{n-2}{4}-c-k}{n-2-4k}$$

which may be simplified, by means of (0.1a), to the following relation:

$$\Delta_1(n,c) = (-1)^{\frac{n-1}{2}} \frac{-2c}{\frac{1+n}{4}-2c}\binom{\frac{1+n}{4}-2c}{1+n} \tag{3.13a}$$

$$- (-1)^{\frac{n-1}{2}} \lozenge_3(n-2,-c), \quad n \equiv 1 \pmod 2 \tag{3.13b}$$

$$= \Delta_1(n,-c), \quad n \equiv 3 \pmod 4, \tag{3.13c}$$

where the last line is derived by reversing the summation order.

When $n \equiv 0 \pmod 2$, it is easy to check from Table 2 that

$$\lambda_1(n,k) + \lambda_1(n, 1+n-k) = -2(-1)^{\frac{(n-k)(k-1)}{2}}.$$

Then the combination of (3.12b) and its reversal enables us to express

$$\Delta_1(n,c) = \sum_k \frac{c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{1+n-k}{4}-c}\binom{\frac{1+n-k}{4}-c}{1+n-k}(-1)^{\frac{(n-k)(k-1)}{2}}$$

$$= \sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{1+n-k}{4}-c}\binom{\frac{1+n-k}{4}-c}{1+n-k} - 4\sum_{k\equiv 1 \pmod 4} \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{1+n-k}{4}-c}\binom{\frac{1+n-k}{4}-c}{1+n-k}.$$

Applying (0.1a) to the penultimate sum, we get

$$\Delta_1(n,c) = -\lozenge_1(n,-c) + \frac{-2c}{\frac{1+n}{4}-2c}\binom{\frac{1+n}{4}-2c}{1+n} \tag{3.14a}$$

$$= -\Delta_1(n,-c), \quad n \equiv 0 \pmod 2. \tag{3.14b}$$

Both relations (3.13) and (3.14) may be restated as

$$\Delta_1(n,c) + \Delta_1(n,-c)\varepsilon(1,n) = 0, \quad n \not\equiv 1 \pmod 4$$

$$\lozenge_1(n,c) + \lozenge_3(n-2,-c): \qquad n \equiv 1 \pmod 4$$

$$= \frac{2c}{\frac{1+n}{4}+2c}\binom{\frac{1+n}{4}+2c}{1+n} + \frac{-2c}{\frac{1+n}{4}-2c}\binom{\frac{1+n}{4}-2c}{1+n}$$

which, in view of (3.12a), confirms the case $m \equiv 1 \pmod 4$ of Theorem 3.3 with replacements $k \to k+p$ and $n \to n+4p$.

Case 3: $m \equiv 2 \pmod 4$. Using Lemma 3.1 and (3.4e)-(3.4f), perform the formal manipulation on generating functions:

$$-W^{8c}(\tau) + W^{4c}(\tau) \times \{W^{4c}(\tau) + W^{4c}(-\tau) - W^{4c}(i\tau) - W^{4c}(-i\tau)\}$$
$$= W^{-4c}(i\tau)W^{-4c}(-i\tau) - W^{-4c}(-\tau)W^{-4c}(i\tau) - W^{-4c}(-\tau)W^{-4c}(-i\tau).$$

The coefficients of τ^{2+n} in the formal power series expansions lead us to the following binomial convolutions,

$$\Delta_2(n,c) \stackrel{\text{def}}{=} \lozenge_2(n,c) - \frac{2c}{2c+\frac{2+n}{4}}\binom{2c+\frac{2+n}{4}}{2+n} \tag{3.15a}$$

$$= \sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{2+n-k}{4}-c}\binom{\frac{2+n-k}{4}-c}{2+n-k}\lambda_2(n,k), \tag{3.15b}$$

where

$$\lambda_2(n,k) = (-1)^n\{i^{2+k} + i^{2+3k} + i^{2+n+2k}\}$$

whose values are displayed in Table 3.

For $n \equiv 0 \pmod 2$, Table 3 suggests that we reformulate (3.15b) as

$$\Delta_2(n,c) = (-1)^{n/2}\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{2+n-k}{4}-c}\binom{\frac{2+n-k}{4}-c}{2+n-k}$$
$$-(-1)^{n/2}\sum_k \frac{-4c}{\frac{1}{2}-c+k}\binom{\frac{1}{2}-c+k}{2+4k}\frac{-c}{\frac{n}{4}-c-k}\binom{\frac{n}{4}-c-k}{n-4k},$$

which may be simplified, by means of (0.1a), to the following relation:

$$\Delta_2(n,c) = -(-1)^{n/2}\Diamond_2(n,-c) + (-1)^{n/2}\frac{-2c}{\frac{2+n}{4}-2c}\binom{\frac{2+n}{4}-2c}{2+n} \tag{3.16a}$$

$$= -(-1)^{n/2}\Delta_2(n,-c), \quad n \equiv 0 \pmod 2. \tag{3.16b}$$

TABLE 3. Values of $\lambda_2(n,k)$

$k \backslash n$	0	1	2	3
0	-3	$i+2$	-1	$2-i$
1	1	$-i$	-1	i
2	1	$i-2$	3	$-2-i$
3	1	$-i$	-1	i

While $n \equiv 1 \pmod 2$, it is easy to check from Table 3 above that

$$\lambda_2(n,k) + \lambda_2(n, 2+n-k) = 2(-1)^{\frac{k(n+k)}{2}}.$$

Then the combination of (3.12b) and its reversal enables us to state

$$\Delta_2(n,c) = \sum_k \frac{c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{2+n-k}{4}-c}\binom{\frac{2+n-k}{4}-c}{2+n-k}(-1)^{\frac{k(n+k)}{2}}$$
$$= \sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{2+n-k}{4}-c}\binom{\frac{2+n-k}{4}-c}{2+n-k} - 4\sum_{k \equiv 2 \pmod 4} \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{2+n-k}{4}-c}\binom{\frac{2+n-k}{4}-c}{2+n-k}.$$

Applying (0.1a) to the penultimate sum, we get

$$\Delta_2(n,c) = -\Diamond_2(n,-c) + \frac{-2c}{\frac{2+n}{4}-2c}\binom{\frac{2+n}{4}-2c}{2+n} \tag{3.17a}$$

$$= -\Delta_2(n,-c), \quad n \equiv 1 \pmod 2. \tag{3.17b}$$

Both relations (3.16) and (3.17) may be written as the single relation

$$\Delta_2(n,c) + \Delta_2(n,-c)\varepsilon(2,n) = 0,$$

which confirms, in view of (3.15a), the case $m \equiv 2 \pmod 4$ of Theorem 3.3 with replacements $k \to k+p$ and $n \to n+4p$.

Case 4: $m \equiv 3 \pmod 4$. Finally, from Lemma 3.1 and (3.4g)-(3.4h), we get the following functional equation:

$$-W^{8c}(\tau)+W^{4c}(\tau)\times\{W^{4c}(\tau)-W^{4c}(-\tau)+iW^{4c}(i\tau)-iW^{4c}(-i\tau)\}$$
$$=W^{-4c}(i\tau)W^{-4c}(-i\tau)-iW^{-4c}(-\tau)W^{-4c}(i\tau)+iW^{-4c}(-\tau)W^{-4c}(-i\tau).$$

The coefficients of τ^{3+n} in their formal power series expansions leads us to the following binomial convolutions,

$$\Delta_3(n,c)\stackrel{\text{def}}{=}\lozenge_3(n,c)-\frac{2c}{2c+\frac{3+n}{4}}\binom{2c+\frac{3+n}{4}}{3+n} \tag{3.18a}$$

$$=\sum_k\frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{3+n-k}{4}-c}\binom{\frac{3+n-k}{4}-c}{3+n-k}\lambda_3(n,k), \tag{3.18b}$$

where

$$\lambda_3(n,k)=(-1)^n\{i^{3+k}+i^{1+3k}+i^{3+n+2k}\}$$

whose values are displayed in Table 4.

TABLE 4. Values of $\lambda_3(n,k)$

$k \backslash n$	0	1	2	3
0	$-i$	-1	i	1
1	$i+2$	-1	$2-i$	-3
2	$-i$	-1	i	1
3	$i-2$	3	$-2-i$	1

For $n\equiv 1\pmod 2$, Table 4 suggests that we rewrite (3.18b) as

$$\Delta_3(n,c)=(-1)^{\frac{n-1}{2}}\sum_k\frac{-4c}{\frac{1}{4}-c+k}\binom{\frac{1}{4}-c+k}{1+4k}\frac{-c}{\frac{2+n}{4}-c-k}\binom{\frac{2+n}{4}-c-k}{2+n-4k}$$

$$-(-1)^{\frac{n-1}{2}}\sum_k\frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{3+n-k}{4}-c}\binom{\frac{3+n-k}{4}-c}{3+n-k},$$

which may be simplified, by means of (0.1a), to the following relation,

$$\Delta_3(n,c)=(-1)^{\frac{n+1}{2}}\frac{-2c}{\frac{3+n}{4}-2c}\binom{\frac{3+n}{4}-2c}{3+n} \tag{3.19a}$$

$$+(-1)^{\frac{n-1}{2}}\lozenge_1(2+n,-c),\quad n\equiv 1\ (\mathrm{mod}\,2) \tag{3.19b}$$

$$=(-1)^{\frac{n-1}{2}}\Delta_3(n,-c),\quad n\equiv 1\ (\mathrm{mod}\,4), \tag{3.19c}$$

where the last line is derived by reversing the summation order.

While $n\equiv 0\pmod 2$, it is easy to check from Table 4 that

$$\lambda_3(n,k)+\lambda_3(n,3+n-k)=-2(-1)^{\frac{(n-k)(k+1)}{2}}.$$

Then, the combination of (3.18b) and its reversal leads us to

$$\Delta_3(n,c)=\sum_k\frac{c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{3+n-k}{4}-c}\binom{\frac{3+n-k}{4}-c}{3+n-k}(-1)^{\frac{(n-k)(k+1)}{2}}$$

$$=\sum_k \frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{3+n-k}{4}-c}\binom{\frac{3+n-k}{4}-c}{3+n-k}-4\sum_{k\equiv 3\,(\mathrm{mod}\,4)}\frac{-c}{\frac{k}{4}-c}\binom{\frac{k}{4}-c}{k}\frac{-c}{\frac{3+n-k}{4}-c}\binom{\frac{3+n-k}{4}-c}{3+n-k}.$$

Applying (0.1a) to the penultimate sum, we get

$$\Delta_3(n,c)=-\lozenge_3(n,-c)+\frac{-2c}{\frac{3+n}{4}-2c}\binom{\frac{3+n}{4}-2c}{3+n} \tag{3.20a}$$

$$=-\Delta_3(n,-c),\quad n\equiv 0 \pmod 2. \tag{3.20b}$$

Both relations (3.19) and (3.20) may be reproduced as

$$\Delta_3(n,c)+\Delta_3(n,-c)\varepsilon(3,n)=0,\quad n\not\equiv 3 \pmod 4,$$

$$\lozenge_3(n,c)+\lozenge_1(2+n,-c):\quad n\equiv 3 \pmod 4$$

$$=\frac{2c}{\frac{3+n}{4}+2c}\binom{\frac{3+n}{4}+2c}{3+n}+\frac{-2c}{\frac{3+n}{4}-2c}\binom{\frac{3+n}{4}-2c}{3+n},$$

which confirms, in view of (3.18a), the case $m\equiv 3$ (mod 4) of Theorem 3.3 with replacements $k\to k+p$ and $n\to n+4p$.

Therefore, the proof of Theorem 3.3 is complete. □

Remark: During the 100th anniversary of Tricomi (October 1997, Rome), Richard Askey suggested that the author try another approach to the binomial identities stated in Theorem 1.3. This may be presented as follows:

Letting $\beta=1/2$ in (0.1b), we obtain

$$\sum\frac{a}{a+k/2}\binom{a+k/2}{k}\binom{c-k/2}{n-k}=\binom{a+c}{n}. \tag{3.21a}$$

By means of

$$\frac{a}{a+k/2}\binom{a+k/2}{k}=(-1)^k\frac{-a}{-a+k/2}\binom{-a+k/2}{k},$$

we can reformulate (3.21a) as

$$\sum_{k=0}^{n}(-1)^k\frac{a}{a+k/2}\binom{a+k/2}{k}\binom{c-k/2}{n-k}=\binom{c-a}{n}. \tag{3.21b}$$

Then identities (1.5a) and (1.5b) follow directly from the combinations of (3.21a) and (3.21b). Two other identities, stated in (1.6a)-(1.6b) and (1.6c)-(1.6d), may be derived similarly from (0.1a). The details are left to the reader.

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