# ELEMENTARY PROBLEMS AND SOLUTIONS

## *Edited by*
## Russ Euler and Jawad Sadek

*Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.*

*If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.*

*Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2002. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".*

### BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

### PROBLEMS PROPOSED IN THIS ISSUE

**B-930** ***Proposed by José Luis Díaz and Juan José Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain***

Let $n \geq 0$ be a nonnegative integer. Prove that $F_n^{L_n} L_n^{F_n} \leq (F_{n+1}^{F_{n+1}})^2$. When does equality occur?

**B-931** ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan***

Prove that $\gcd(L_n, F_{n+1}) = 1$ for all $n \geq 0$.

**B-932** ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan***

Prove that

**A)** $\dfrac{F_2 F_4 \ldots F_{2n}}{F_1 F_3 \ldots F_{2n+1}} < \dfrac{1}{\sqrt{F_{2n+1}}}$ for all $n \geq 1$

and

**B)** $\displaystyle\sum_{k=1}^{\infty} \frac{F_2 F_4 \ldots F_{2k}}{F_1 F_3 \ldots F_{2k+1}}$ converges.

**B-933** ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan***

Prove that $F_n^{F_{n+1}} > F_{n+1}^{F_n}$ for all $n \geq 4$.

**B-934** ***Proposed by N. Gauthier, Royal Military College of Canada***

Prove that

$$2\sum_{n=1}^{m}\sin^2\left(\frac{\pi}{2}\frac{F_{m+1}}{F_mF_nF_{n+1}}\right)\sin\left(\pi\frac{F_nF_{m+1}}{F_mF_{n+1}}\right)=\sum_{n=1}^{m}(-1)^n\sin\left(\pi\frac{F_{m+1}}{F_mF_nF_{n+1}}\right)\cos\left(\pi\frac{F_nF_{m+1}}{F_{n+1}F_m}\right),$$

where $m$ is a positive integer.

**Note:** *The Elementary Problems Editor, Dr. Russ Euler, is in need of more easy yet elegant and nonroutine problems. Due to space limitations, brevity is also important.*

## SOLUTIONS

### Subscript Is Power

**B-916** ***Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (Vol. 39, no. 2, May 2001)***

Determine the value of

$$\prod_{k=0}^{n}(L_{2\cdot3^k}-1).$$

***Solution 1 by H.-J. Seiffert, Berlin, Germany***

In equation (2.6) of [1], it was shown that

$$\prod_{k=1}^{n}(L_{2\cdot3^k}-1)=\frac{1}{2}F_{3^{n+1}}.$$

Multiplying by $L_2-1=2$ gives

$$\prod_{k=0}^{n}(L_{2\cdot3^k}-1)=F_{3^{n+1}}.$$

**Reference**

1. P. Filipponi. "On the Fibonacci Numbers whose Subscript Is a Power." *The Fibonacci Quarterly* **34.3** (1996):271-76.

***Solution 2 by Paul S. Bruckman, Sacramento, CA***

For brevity, write $3^k=u$, $A_k\equiv F_u$. Note that $A_{k+1}/A_k=F_{3u}/F_u=(\alpha^{3u}-\beta^{3u})/(\alpha^u-\beta^u)=\alpha^{2u}+\alpha^u\beta^u+\beta^{3u}=L_{2u}-1$, since $u$ is odd. Let $P_n$ denote the given product. Then

$$P_n=\prod_{k=0}^{n}(A_{k+1}/A_k),$$

a telescoping product that reduces to $A_{n+1}/A_0$. Therefore, since $A_0=F_1=1$, it follows that $P_n=F_v$, where $v=3^{n+1}$.

***Also solved by Brian Beasley, Kenneth Davenport, L. A. G. Dresel, Steve Edwards, Ovidiu Furdui, Russell Hendel, Walther Janous, Reiner Martin, Don Redmond, Maitland Rose, Jaroslav Seibert, and the proposer.***

*The solvers all used almost the same induction argument to show the result.*

## A Two Sum Problem

**B-917** ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain (Vol. 39, no. 2, May 2001)***

Find the following sums:

$$(a)\ \sum_{n\geq 0} \frac{1+L_{n+1}}{L_n L_{n+2}}, \qquad (b)\ \sum_{n\geq 0} \frac{L_{n-1}L_{n+2}}{L_n^2 L_{n+1}^2},$$

where $L_k$ is the $k^{\text{th}}$ Lucas number.

***Solution by Reiner Martin, New York, NY***

Note that

$$\sum_{n=0}^{m} \frac{1}{L_n L_{n+2}} = \sum_{n=0}^{m}\left(\frac{1}{L_n L_{n+1}} - \frac{1}{L_{n+1}L_{n+2}}\right) = \frac{1}{2} - \frac{1}{L_{m+1}L_{m+2}}$$

and

$$\sum_{n=0}^{m} \frac{L_{n+1}}{L_n L_{n+2}} = \sum_{n=0}^{m}\left(\frac{1}{L_n} - \frac{1}{L_{n+2}}\right) = \frac{3}{2} - \frac{1}{L_{m+1}} - \frac{1}{L_{m+2}}.$$

So we can evaluate (a) as

$$\sum_{n\geq 0} \frac{1+L_{n+1}}{L_n L_{n+2}} = \frac{1}{2} + \frac{3}{2} = 2.$$

To find (b), observe that

$$\sum_{n=0}^{m} \frac{L_{n-1}L_{n+2}}{L_n^2 L_{n+1}^2} = \sum_{n=0}^{m}\left(\frac{1}{L_n^2} - \frac{1}{L_{n+1}^2}\right) = \frac{1}{4} - \frac{1}{L_{m+1}^2},$$

which converges to 1/4.

***Also solved by Paul S. Bruckman, L. A. G. Dresel, Steve Edwards, Ovidiu Furdui, Russell Hendel, Walther Janous, Jaroslav Seibert, H.-J. Seiffert, and the proposer.***

## Divisible or Not Divisible; That Is, by 2

**B-918** ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India (Vol. 39, no. 2, May 2001)***

Let $i$ and $j$ be positive integers such that $1 \leq j \leq i$. Let

$$T(i, j) = F_j F_{i-j+1} + F_j F_{i-j+2} + F_{j+1} F_{i-j+1}.$$

Determine whether or not

$$\underset{j}{\text{maximum}}\ T(i, j) - \underset{j}{\text{minimum}}\ T(i, j)$$

is divisible by 2 for all $i \geq 3$.

***Solution by L. A. G. Dresel, Reading, England***

Since $F_j + F_{j+1} = F_{j+2}$, we have $T(i, j) = F_{j+2}F_{i-j+1} + F_j F_{i-j+2}$, and therefore

$$T(i, j+1) = F_{j+3}F_{i-j} + F_{j+1}F_{i-j+1} \equiv F_j F_{i-j} + F_{j+1}F_{i-j+1} \pmod 2,$$

since $F_j \equiv F_{j+3} \pmod 2$. Hence,

$$T(i,j)-T(i,j+1) \equiv (F_{j+2}-F_{j+1})F_{i-j+1}+F_j(F_{i-j+2}-F_{i-j}) = 2F_jF_{i-j+1} \equiv 0 \pmod 2.$$

It follows that, for a given $i$, $T(i,j)$ modulo 2 is independent of $j$, and therefore the difference between the given maximum and minimum is divisible by 2 for each $i$. We note that

$$T(i,j) \equiv T(i,1) \equiv F_{i+1} \pmod 2$$

for all $j$.

***Also solved by Paul S. Bruckman, Walther Janous, Reiner Martin, Jaroslav Seibert, H.-J. Seiffert, and the proposer.***

## A Prime Equation

**B-919** ***Proposed by Richard André-Jeannin, Cosnes et Romain, France (Vol. 39, no. 2, May 2001)***

Solve the equation $L_nF_{n+1} = p^m(p^m-1)$, where $m$ and $n$ are natural numbers and $p$ is a prime number.

***Solution by Jaroslav Seibert, Hradec Králové, The Czech Republic***

First, we will prove that $L_n$ and $F_{n+1}$ are relatively prime numbers for each natural number $n$. Suppose there exists a prime $q$ such that it divides the numbers $L_n$ and $F_{n+1}$ for some $n$. It is known that $L_nF_{n+1} = F_{2n+1}+(-1)^n = F_{n+1}L_{n+1}-F_nL_n+(-1)^n$ (see S. Vajda, *Fibonacci and Lucas Numbers & the Golden Section*, pp. 25, 36). Since the prime $q$ divides $F_{n+1}L_{n+1}-F_nL_n$, it cannot divide $F_nL_{n+1}$, which is a contradiction. Further, it is easy to see that $F_{n+1} \le L_n \le 2F_{n+1}$, because $L_n = F_{n+1}+F_{n-1}$. To solve the given equation, it must be $F_{n+1} = p^m-1$ and $L_n = p^m$, which means $L_n - F_{n+1} = 1$. Then

$$\alpha^n+\beta^n-\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} = 1,$$

$$\alpha^{n+1}-\alpha^n\beta+\alpha\beta^n-\beta^{n+1}-\alpha^{n+1}+\beta^{n+1} = \alpha-\beta,$$

$$\alpha^{n-1}-\beta^{n-1} = \alpha-\beta,$$

$$\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} = 1.$$

But $F_{n-1} = 1$ holds only for $n-1=1$, $n-1=2$, and $n-1=-1$ if we admit Fibonacci numbers with negative indices.

The given equation has only two solutions for natural numbers $n$, $m$. Concretely, if $n=2$, $m=1$, $p=3$, then $L_2F_3 = 3^1(3^1-1)$; if $n=3$, $m=2$, $p=2$, then $L_3F_4 = 2^2(2^2-1)$. In addition, if $n=0$, $m=1$, $p=2$, the equality $L_0F_1 = 2^1(2^1-1)$ also holds.

*Florian Luca commented that the equation $L_nF_{n+1} = x(x-1)$, where $n$ and $x$ are nonnegative integers is a more general equation. He proved in one of his forthcoming papers that it has solutions at $n = 0$, 2, and 3 as well.*

***Also solved by Brian Beasley, Paul S. Bruckman, L. A. G. Dresel, Ovidiu Fordui, Walther Janous, Florian Luca, H.-J. Seiffert, and the proposer.***

## A Trigonometric Sum

**B-920** ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 39, no. 2, May 2001)***

Prove that

$$\sum_{n=1}^{\infty} \sin\left(\frac{p\pi}{2}\cdot\frac{F_{n-1}}{F_nF_{n+1}}\right)\cos\left(\frac{p\pi}{2}\cdot\frac{F_{n+2}}{F_nF_{n+1}}\right) = 0$$

for $p$ an arbitrary integer.

***Solution by Steve Edwards, Marietta, GA***

$$\begin{aligned}\sum_{n=1}^{\infty} \sin\left(\frac{p\pi}{2}\cdot\frac{F_{n-1}}{F_nF_{n+1}}\right)\cos\left(\frac{p\pi}{2}\cdot\frac{F_{n+2}}{F_nF_{n+1}}\right) &= \sum_{n=1}^{\infty} \sin\left(\frac{p\pi}{2}\cdot\frac{F_{n+1}-F_n}{F_nF_{n+1}}\right)\cos\left(\frac{p\pi}{2}\cdot\frac{F_{n+1}+F_n}{F_nF_{n+1}}\right)\\ &= \sum_{n=1}^{\infty} \sin\left(\frac{p\pi}{2}\cdot\left(\frac{1}{F_n}-\frac{1}{F_{n+1}}\right)\right)\cos\left(\frac{p\pi}{2}\cdot\left(\frac{1}{F_n}+\frac{1}{F_{n+1}}\right)\right).\end{aligned}$$

Now we can use the trig identity $\sin\frac{a-b}{2}\cos\frac{a+b}{2} = \frac{1}{2}(\sin a - \sin b)$ to get

$$\frac{1}{2}\sum_{n=1}^{\infty} \sin\frac{p\pi}{F_n} - \sin\frac{p\pi}{F_{n+1}}.$$

But this is a telescoping sum. Since $\frac{1}{F_n} \to 0$ as $n \to \infty$, only $\frac{1}{2}\sin\frac{p\pi}{F_1}$ survives, and this is 0 for any integer $p$.

*Some solvers noted that the result is true for p any arbitrary complex number.*

***Also solved by Paul S. Bruckman, Kenneth B. Davenport, L. A. G. Dresel, Ovidiu Fordui, Toufik Mansour, Reiner Martin, Maitland Rose, Jaroslav Seibert, H.-J. Seiffert, and the proposer.***

We wish to belatedly acknowledge the solution to Problem B-901 by Charles K. Cook, and the solution to Problem B-911 by Lake Superior State University Problem Solving Group.

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