AN ALTERNATE PROOF OF A THEOREM OF J. EWELL

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Let $t_r(n)$ denote the number of representations of n as a sum of r triangular numbers. In [1], J. Ewell derived a sextuple product identity, one of whose consequences is

Theorem 1: For each integer $n \geq 0$,

$$t_4(n) = \sigma(2n+1)$$

where σ denotes the arithmetical sum-of-divisors function.

In this note we present an alternate proof of Theorem 1.

Proof: Clearly, n is a sum of four triangular numbers if and only if $8n+4$ is a sum of the squares of four odd positive integers. Let $r_4(n)$ denote the number of representations of n as a sum of four squares, while $s_4(n)$ denotes the number of representations of n as the sum of four odd squares. An elementary argument shows that, if $8n+4$ is a sum of four squares, then these squares must all have the same parity. It is easily seen that

$$8n+4 = \sum_{i=1}^{4}(2b_i)^2$$

if and only if

$$2n+1 = \sum_{i=1}^{4} b_i^2.$$

Therefore, we have

$$\begin{aligned} s_4(8n+4) &= r_4(8n+4) - r_4(2n+1) = 8\left(\sum\{d : d \mid (8n+4), 4 \nmid d\} - \sum\{d : d \mid (2n+1)\}\right) \\ &= 8(\sigma(4n+2) - \sigma(2n+1)) = 16\sigma(2n+1), \end{aligned}$$

according to a well-known formula of Jacobi ([2], Theorem 386, p. 312). Therefore, the number of representations of $8n+4$ as the sum of the squares of four odd *positive* integers is

$$\frac{1}{16}s_4(8n+4) = \sigma(2n+1)$$

from which the conclusion follows.

REFERENCES

1. J. A. Ewell. "Arithmetical Consequences of a Sextuple Product Identity." *Rocky Mountain J. Math.* **25** (1995):1287-93.
2. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. Oxford: Oxford University Press, 1960.

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