ON THE NUMBER OF PARTITIONS INTO AN EVEN AND ODD NUMBER OF PARTS

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INTRODUCTION

Let $q_i^e(n)$, $q_i^o(n)$ denote, respectively, the number of partitions into n evenly many, oddly many parts, with each part occurring at most i times. Let $\Delta_i(n) = q_i^e(n) - q_i^o(n)$. Let $\omega(j) = j(3j-1)/2$. It is well known that

$$\Delta_1(n) = \begin{cases} (-1)^n & \text{if } n = \omega(\pm j), \\ 0 & \text{otherwise.} \end{cases}$$

Formulas for $\Delta_i(n)$ were obtained by Hickerson [2] in the cases $i = 3$, i even; by Alder & Muwafi [1] in the cases $i = 5, 7$; by Hickerson [3] for i odd. In this note, we present a simpler formula for $\Delta_i(n)$, where i is odd, than that given in [3]. As a consequence, we obtain two apparently new recurrences concerning $q(n)$.

Remark: Note that, if f denotes any partition function, then we define $f(\alpha) = 0$ if α is not a nonnegative integer.

PRELIMINARIES

Definition 1: If $r \geq 2$, let $b_r(n)$ denote the number of r-regular partitions of n, i.e., the number of partitions of n into parts not divisible by r, or equivalently, the number of partitions of n such that each part occurs less than r times.

Let $x \in C$, $|x| < 1$. Then we have

$$\prod_{n\geq 1}(1-x^n) = 1 + \sum_{k\geq 1}(-1)^k(x^{\omega(k)} + x^{\omega(-k)}), \tag{1}$$

$$\sum_{n\geq 0} b_r(n)x^n = \prod_{n\geq 1}\frac{1-x^{rn}}{1-x^n}, \tag{2}$$

$$\sum_{n\geq 0}\Delta_i(n)x^n = \prod_{n\geq 1}\frac{1+(-1)^i x^{(i+1)n}}{1+x^n}, \tag{3}$$

$$\Delta_3(n) = \begin{cases} (-1)^n & \text{if } n = j(j+1)/2, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Theorem 1: If $r \geq 2$, then

$$\Delta_{2r-1}(n) = b_r\left(\frac{n}{2}\right) + \sum_{k\leq 1}(-1^k)\left\{\left(b_r\left(\frac{n-\omega(k)}{2}\right) + b_r\left(\frac{n-\omega(-k)}{2}\right)\right)\right\}.$$

Proof: Invoking (3), (2), and (1), we have

$$\begin{aligned}\sum_{n\le 0}\Delta_{2r-1}(n)x^n &= \prod_{n\ge1}\frac{1-x^{2rn}}{1+x^n}\\ &= \prod_{n\ge1}\frac{1-x^{2rn}}{1-x^{2n}}\prod_{n\ge1}(1-x^n)=\left(\sum_{n\ge0}b_r\left(\frac{n}{2}\right)x^n\right)\prod_{n\ge1}(1-x^n)\\ &= \sum_{n\ge0}\left(b_r\left(\frac{n}{2}\right)+\sum_{k\ge1}(-1)^k\left\{\left(b_r\left(\frac{n-\omega(k)}{2}\right)+b_r\left(\frac{n-\omega(-k)}{2}\right)\right)\right\}\right)x^n.\end{aligned}$$

The conclusion now follows by matching coefficients of like powers of x.

Theorem 2:

(a) $$q(n)+\sum_{k\ge1}(-1)^k\left\{q\left(n-\frac{\omega(k)}{2}\right)+q\left(n-\frac{\omega(-k)}{2}\right)\right\}=\begin{cases}1 & \text{if } n=j(j+1)/4,\\ 0 & \text{otherwise.}\end{cases}$$

(b) $$q(n)+\sum_{k\ge2}(-1)^{k-1}\left\{q\left(n+\frac{1-\omega(k)}{2}\right)+q\left(n+\frac{1-\omega(-k)}{2}\right)\right\}=\begin{cases}1 & \text{if } n=j(j+3)/4,\\ 0 & \text{otherwise.}\end{cases}$$

Proof: Apply Theorem 1 with $r=2$, noting that $b_2(n)=q(n)$. This yields

$$q\left(\frac{n}{2}\right)+\sum_{k\ge1}(-1)^k\left\{q\left(\frac{n-\omega(k)}{2}\right)+q\left(\frac{n-\omega(-k)}{2}\right)\right\}=\Delta_3(n).$$

If we invoke (4) and replace n by $2n$, we get (a); similarly, if we replace n by $2n+1$, we get (b).

Since it is easily seen that $2|\omega(k)$ iff $k\equiv 0,3 \pmod 4$, we may rewrite Theorem 2 in a fraction-free form as follows.

Theorem 2*:

(a) $$\begin{aligned}&q(n)-q(n-1)+\sum_{i\ge1}(q(n-(4i-1)(3i-1))+q(n-(n-(4i+1)(3i+1)))\\ &-\sum_{i\ge1}(q(n-i(12i-1))+q(n-i(12i+1)))=\begin{cases}1 & \text{if } n=j(j+1)/4,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

(b) $$\begin{aligned}&q(n)+\sum_{i\ge1}q(n-i(12i-5))+q(n-i(12i+5))-\sum_{i\ge1}(q(n-(4i-3)(3i-1))\\ &+q(n-(4i-1)(3i-2)))=\begin{cases}1 & \text{if } n=j(j+3)/4,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

REFERENCES

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