ON THE RESOLUTION OF THE EQUATIONS $U_n = \binom{x}{3}$ AND $V_n = \binom{x}{3}$

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1. INTRODUCTION

The purpose of the present paper is to prove that there are finitely many binomial coefficients of the form $\binom{x}{3}$ in certain binary recurrences, and give a simple method for the determination of these coefficients. We illustrate the method by the Fibonacci, the Lucas, and the Pell sequences. First, we transform both of the title equations into two elliptic equations and apply a theorem of Mordell [10], [11] to them. (Later, Siegel [16] generalized Mordell's result, and in 1968 Baker [1] gave its effective version.) After showing the finiteness, we use the program package SIMATH [15] which is a computer algebra system, especially useful for number theoretic purposes, and is able to find all the integer points on the corresponding elliptic curves. The algorithms of SIMATH are based on some deep results of Gebel, Pethö, and Zimmer [5].

Before going into detail, we present a short historical survey. Several authors have investigated the occurrence of special figurate numbers in the second-order linear recurrences. One such problem is, for example, to determine which Fibonacci numbers are square. Cohn [2], [3] and Wyler [18], applying elementary methods, proved independently that the only square Fibonacci numbers are $F_0 = 0$, $F_1 = F_2 = 1$, and $F_{12} = 144$. A similar result for the Lucas numbers was obtained by Cohn [4]: if $L_n = x^2$, then $n = 1$ or $n = 3$. London and Finkelstein [6] established full Fibonacci cubes. Pethö [12] gave a new proof of the theorem of London and Finkelstein, applying the Gel'fond-Baker method and computer investigations. Later Pethö found all the fifth-power Fibonacci numbers [14], and all the perfect powers in the Pell sequence [13].

Another special interest was to determine the triangular numbers $T_x = \frac{x(x+1)}{2}$ in certain recurrences. Hoggatt conjectured that there are only five triangular Fibonacci numbers. This problem was originally posed in 1963 by Tallman [17] in *The Fibonacci Quarterly*. In 1989 Ming [8] proved Hoggatt's conjecture by showing that the only Fibonacci numbers that are triangular are $F_0 = 0$, $F_1 = F_2 = 1$, $F_4 = 3$, $F_8 - 21$, and $F_{10} = 55$. Ming also proved in [9] that the only triangular Lucas numbers are $L_1 = 1$, $L_2 = 3$, and $L_{18} = 5778$. Moreover, the only triangular Pell number is $P_1 = 1$ (see McDaniel [7]).

Since the number T_{x-1} is equal to the binomial coefficient $\binom{x}{2}$, it is natural to ask whether the terms $\binom{x}{3}$ occur in binary recurrences or not. As we will see, the second-order linear recurrences, for instance, the Fibonacci, the Lucas, and the Pell sequences have few such terms.

Now we introduce some notation. Let the sequence $\{U_n\}_{n=0}^{\infty}$ be defined by the initial terms U_0, U_1, and by the recurrence relation

$$U_n = AU_{n-1} + BU_{n-2} \quad (n \geq 2), \tag{1}$$

where $U_0, U_1, A, B \in \mathbf{Z}$ with the conditions $|U_0| + |U_1| > 0$ and $AB \neq 0$. Moreover, let α and β be the roots of the polynomial

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$$p(x) = x^2 - Ax - B, \tag{2}$$

and we denote the discriminant $A^2 + 4B$ of $p(x)$ by D. Suppose $D \neq 0$ (i.e., $\alpha \neq \beta$). Throughout this paper we also assume that $U_0 = 0$ and $U_1 = 1$.

The sequence

$$V_n = AV_{n-1} + BV_{n-2} \quad (n \geq 2), \tag{3}$$

with the initial values $V_0 = 2$ and $V_1 = A$ is the associate sequence of U. The recurrences U and V satisfy the relation $V_n^2 - DU_n^2 = 4(-B)^n$.

Finally, it is even assumed that $|B| = 1$. Then

$$V_n^2 - DU_n^2 = 4(\pm 1)^n = \pm 4. \tag{4}$$

As usual, denote by F_n, L_n, and P_n the n^{th} term of the Fibonacci, the Lucas, and the Pell sequences, respectively.

The following theorems formulate precisely the new results.

Theorem 1: Both the equations $U_n = \binom{x}{3}$ and $V_n = \binom{x}{3}$ have only a finite number of solutions (n, x) in the integers $n \geq 0$ and $x \geq 3$.

Theorem 2: All the integer solutions of the equation

(i) $F_n = \binom{x}{3}$ are $(n, x) = (1, 3)$ and $(2, 3)$,

(ii) $L_n = \binom{x}{3}$ are $(n, x) = (1, 3)$ and $(3, 4)$,

(iii) $P_n = \binom{x}{3}$ is $(n, x) = (1, 3)$.

2. PROOF OF THEOREM 1

Let U and V be binary recurrences specified above. We distinguish two cases.

Case 1. First, we deal with the equation

$$U_n = \binom{x}{3} \tag{5}$$

in the integers n and x. Applying (4) together with $y = V_n$ and $x_1 = x - 1$, we have $U_n = \binom{x_1+1}{3}$ and

$$y^2 - D\left(\frac{x_1^3 - x_1}{6}\right)^2 = \pm 4. \tag{6}$$

Take the 36 times of the equation (6). Let $x_2 = x_1^2$ and $y_1 = 6y$, and using these new variables, from (6) we get

$$y_1^2 = Dx_2^3 - 2Dx_2^2 + Dx_2 \pm 144. \tag{7}$$

Multiplying by $3^6 D^2$ the equation (7) together with $k = 3^3 D y_1$ and $l = 3D(3x_2 - 2)$, it follows that

$$k^2 = l^3 - 27D^2 l + (54D^3 \pm 104976D^2). \tag{8}$$

By a theorem of Mordell [10], [11], it is sufficient to show that the polynomial $u(l) = l^3 - 27D^2 l + (54D^3 \pm 104976D^2)$ has three distinct roots. Suppose the polynomial $u(l)$ has a multiple root $\tilde{l}$. Then $\tilde{l}$ satisfies $u'(l) = 3l^2 - 27D^2 = 0$, i.e., $\tilde{l} = \pm 3D$. Since $u(3D) = \pm 104976D^2$, it follows that $D=0$, which is impossible. Moreover, $u(-3D)=108D^3 \pm 104976D^2$ implies $D=0$ or $D=\pm 972$. But $D \neq 0$, and by $|B|=1$ there are no integers A for which $D=A^2+4B=\pm 972$. Consequently, $u(l)$ has three distinct zeros.

Case 2. The second case consists of the examination of the Diophantine equation

$$V_n = \binom{x}{3} \tag{9}$$

in the integers n and x. Let $y=U_n$ and $x_1 = x-1$. Applying the method step by step as above in Case 1, it leads to the elliptic equation

$$k^2 = l^3 - 27D^2 l + cD^3, \tag{10}$$

where $c=-104922$ if n is even and $c=105030$ otherwise. The polynomial $v(l)=l^3-27D^2l+cD^3$ also has three distinct roots because $v'(l)=3l^2-27D^2$, $\tilde{l}=\pm 3D$, and $v(\pm 3D)=0$ implies $D=0$. Thus, the proof of Theorem 1 is complete. □

3. PROOF OF THEOREM 2

The corresponding elliptic curves of equations (8) and (10) are, in short, Weierstrass normal form, whence, for a given discriminant D, the theorem can be solved by SIMATH.

By (8) and (10), one can compute the coefficients of the elliptic curves in case of the Fibonacci, the Lucas, and the Pell sequences. The calculations are summarized in Table 1, as well as all the integer points belonging to them. Every binary recurrence leads to two elliptic equations because of the even and odd suffixes. For the Fibonacci and Lucas sequences, $D=5$; for the Pell sequence and its associate sequence, $D=8$.

TABLE 1

Equation	*Transformed equations*	*All the integer solutions* (l,k)
$F_n = \binom{x}{3}$	$k^2 = l^3 - 675l + 2631150$	$(15,1620), (-30,1620), (5199,374868), (735,19980), (150,2430), (-129,756)$
$F_n = \binom{x}{3}$	$k^2 = l^3 - 675l - 2617650$	$(150,810), (555,12960), (1014,32238), (195,2160), (451,9424), (4011,254016)$
$L_n = \binom{x}{3}$	$k^2 = l^3 - 675l - 13115250$	no solution
$L_n = \binom{x}{3}$	$k^2 = l^3 - 675l + 13128750$	$(375,8100), (-74,3574), (150,4050), (-201,2268), (2391,116964)$
$P_n = \binom{x}{3}$	$k^2 = l^3 - 1728l + 6746112$	$(-192,0), (24,2592), (-48,2592), (97,2737), (312,6048), (564,13608), (5208,375840)$
$P_n = \binom{x}{3}$	$k^2 = l^3 - 1728l - 6690816$	$(240,2592), (609,14769)$

The last step is to calculate x and y from the solutions (l,k). By the proof of Theorem 1, it follows that $x = 1+\sqrt{(l+6D)/9D}$, $y=k/162D$ in the case of equation (5), and $y=k/162D^2$ in the case of the associate sequence. Except for some values x and y, they are not integers if $x \geq 3$. The exceptions provide all the solutions of equations (8) and (10). Then the proof of Theorem 2 is complete. □

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