THE PROBABILITY THAT k POSITIVE INTEGERS ARE PAIRWISE RELATIVELY PRIME

László Tóth
University of Pécs, Institute of Mathematics and Informatics, Ifjúság u. 6, H-7624 Pécs, Hungary
E-mail: ltoth@math.ttk.pte.hu

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1. INTRODUCTION

In [3], Shonhiwa considered the function

$$G_k(n) = \sum_{\substack{1 \le a_1, \dots, a_k \le n \\ (a_1, \dots, a_k) = 1}} 1,$$

where $k \ge 2$, $n \ge 1$, and asked: "What can be said about this function?" As a partial answer, he showed that

$$G_k(n) = \sum_{j=1}^{n} \sum_{d \mid j} \mu(d) \left[\frac{n}{d}\right]^{k-1},$$

where μ is the Möbius function (see [3], Theorem 4).

There is a more simple formula, namely,

$$G_k(n) = \sum_{j=1}^{n} \mu(j) \left[\frac{n}{j}\right]^{k}, \tag{1}$$

leading to the asymptotic result

$$G_k(n) = \frac{n^k}{\zeta(k)} + \begin{cases} O(n \log n), & \text{if } k = 2, \\ O(n^{k-1}), & \text{if } k \ge 3, \end{cases} \tag{2}$$

where ζ denotes, as usual, the Riemann zeta function. Formulas (1) and (2) are well known (see, e.g., [1]). It follows that

$$\lim_{n \to \infty} \frac{G_k(n)}{n^k} = \frac{1}{\zeta(k)},$$

i.e., the probability that k positive integers chosen at random are relatively prime is $\frac{1}{\zeta(k)}$.

For generalizations of this result, we refer to [2].

Remark 1: A short proof of (1) is as follows: Using the following property of the Möbius function,

$$G_k(n) = \sum_{1 \le a_1, \dots, a_k \le n} \sum_{d \mid (a_1, \dots, a_k)} \mu(d),$$

and denoting $a_j = db_j$, $1 \le j \le k$, we obtain

$$G_k(n) = \sum_{d=1}^{n} \mu(d) \sum_{1 \le b_1, \dots, b_k \le n/d} 1 = \sum_{d=1}^{n} \mu(d) \left[\frac{n}{d}\right]^k.$$

In what follows, we investigate the question: What is the probability A_k that k positive integers are pairwise relatively prime?

For $k=2$ we have, of course, $A_2 = \frac{1}{\zeta(2)} = 0.607\ldots$ and for $k \geq 3$, $A_k < \frac{1}{\zeta(k)}$. Moreover, for large k, $\frac{1}{\zeta(k)}$ is nearly 1 and A_k seems to be nearly 0.

The next Theorem contains an asymptotic formula analogous to (2), giving the exact value of A_k.

2. MAIN RESULTS

Let $k, n, u \geq 1$ and let

$$P_k^{(u)}(n) = \sum_{\substack{1 \leq a_1, \ldots, a_k \leq n \\ (a_i, a_j)=1, i \neq j \\ (a_i, u)=1}} 1$$

be the number of k-tuples $\langle a_1, \ldots, a_k \rangle$ with $1 \leq a_1, \ldots, a_k \leq n$ such that $a_1, \ldots, a_k$ are pairwise relatively prime and each is prime to u.

Our main result is the following

Theorem: For a fixed $k \geq 1$, we have uniformly for n, $u \geq 1$,

$$P_k^{(u)}(n) = A_k f_k(u) n^k + O(\theta(u) n^{k-1} \log^{k-1} n), \tag{3}$$

where

$$A_k = \prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left(1 + \frac{k-1}{p}\right),$$

$$f_k(u) = \prod_{p \mid u} \left(1 - \frac{k}{p+k-1}\right),$$

and $\theta(u)$ is the number of squarefree divisors of u.

Remark 2: Here $f_k(u)$ is a multiplicative function in u.

Corollary 1: The probability that k positive integers are pairwise relatively prime and each is prime to u is

$$\lim_{n \to \infty} \frac{P_k^{(u)}(n)}{n^k} = A_k f_k(u).$$

Corollary 2: ($u = 1$) The probability that k positive integers are pairwise relatively prime is

$$A_k = \prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left(1 + \frac{k-1}{p}\right).$$

3. PROOF OF THE THEOREM

We need the following lemmas.

Lemma 1: For every $k, n, u \geq 1$,

$$P_{k+1}^{(u)}(n) = \sum_{\substack{j=1 \\ (j, u)=1}}^{n} P_k^{(ju)}(n).$$

Proof: From the definition of $P_k^{(u)}(n)$, we immediately have

$$P_{k+1}^{(u)}(n) = \sum_{\substack{a_{k+1}=1 \\ (a_{k+1},u)=1}}^{n} \sum_{\substack{1\le a_1,\dots,a_k\le n \\ (a_i,a_j)=1, i\ne j \\ (a_i,a_{k+1})=1 \\ (a_i,u)=1}} 1 = \sum_{\substack{a_{k+1}=1 \\ (a_{k+1},u)=1}}^{n} P_k^{(ua_{k+1})}(n) = \sum_{\substack{j=1 \\ (j,u)=1}}^{n} P_k^{(ju)}(n).$$

Lemma 2: For every $k, u \ge 1$,

$$f_k(u) = \sum_{d|u} \frac{\mu(d)k^{\omega(d)}}{\alpha_k(d)},$$

where

$$\alpha_k(u) = u\prod_{p|u}\left(1+\frac{k-1}{p}\right)$$

and $\omega(u)$ stands for the number of distinct prime factors of u.

Proof: By the multiplicativity of the involved functions, it is enough to verify for $n = p^a$ a prime power:

$$\sum_{d|p^a} \frac{\mu(d)k^{\omega(d)}}{\alpha_k(d)} = 1 - \frac{k}{p}\left(1+\frac{k-1}{p}\right)^{-1} = 1 - \frac{k}{p+k-1} = f_k(p^a).$$

Note that, for $k = 2$, $\alpha_2(u) = \psi(u)$ is the Dedekind function.

Lemma 3: For $k \ge 1$, let $\tau_k(n)$ denote, as usual, the number of ordered k-tuples $\langle a_1, \dots, a_k\rangle$ of positive integers such that $n = a_1 \cdot \dots \cdot a_k$. Then

(a)
$$\sum_{n\le x} \frac{\tau_k(n)}{n} = O(\log^k x), \tag{4}$$

(b)
$$\sum_{n>x} \frac{\tau_k(n)}{n^2} = O\left(\frac{\log^{k-1} x}{x}\right). \tag{5}$$

Proof:

(a) Apply the familiar result $\sum_{n\le x} \tau_k(n) = O(x\log^{k-1} x)$ and partial summation.

(b) By induction on k. For $k = 1$, $\tau_1(n) = 1$, $n \ge 1$, and

$$\sum_{n\le x} \frac{1}{n^2} = \zeta(2) + O\left(\frac{1}{x}\right) \tag{6}$$

is well known. Suppose that

$$\sum_{n\le x} \frac{\tau_k(n)}{n^2} = \zeta(2)^k + O\left(\frac{\log^{k-1} x}{x}\right).$$

Then, from the identity $\tau_{k+1}(n) = \sum_{d|n} \tau_k(d)$, we obtain

$$\begin{aligned}\sum_{n\le x} \frac{\tau_{k+1}(n)}{n^2} &= \sum_{de\le x} \frac{\tau_k(e)}{d^2e^2} = \sum_{d\le x}\frac{1}{d^2} \sum_{e\le x/d} \frac{\tau_k(e)}{e^2} \\ &= \sum_{d\le x} \frac{1}{d^2}\left(\zeta(2)^k + O\left(\left(\frac{x}{d}\right)^{-1} \log^{k-1}\frac{x}{d}\right)\right)\end{aligned}$$

$$= \zeta(2)^k \sum_{d \le x} \frac{1}{d^2} + O\left(\frac{\log^{k-1} x}{x} \sum_{d \le x} \frac{1}{d}\right)$$

$$= \zeta(2)^k \left(\zeta(2) + O\left(\frac{1}{x}\right)\right) + O\left(\frac{\log^{k-1} x}{x} \log x\right)$$

by (6), and we get the desired result (5).

Now, for the proof of the Theorem, we use induction on k. For $k = 1$, we have the Legendre function

$$P_1^{(u)}(n) = \sum_{\substack{1 \le a \le n \\ (a,u)=1}} 1 = \sum_{a=1}^{n} \sum_{d \mid (a,u)} \mu(d) = \sum_{a=1}^{n} \sum_{\substack{d \mid a \\ d \mid u}} \mu(d)$$

$$= \sum_{d \mid u} \mu(d) \sum_{1 \le j \le n/d} 1 = \sum_{d \mid u} \mu(d) \left[\frac{n}{d}\right] = \sum_{d \mid u} \mu(d) \left(\frac{n}{d} + O(1)\right)$$

$$= n \sum_{d \mid u} \frac{\mu(d)}{d} + O\left(\sum_{d \mid u} \mu^2(d)\right).$$

Hence,

$$P_1^{(u)}(n) = \sum_{\substack{a=1 \\ (a,u)=1}}^{n} 1 = n \frac{\phi(u)}{u} + O(\theta(u)) \tag{7}$$

and (3) is true for $k = 1$ with $A_1 = 1$, $f_1(u) = \frac{\phi(u)}{u}$, ϕ denoting the Euler function.

Suppose that (3) is valid for k and prove it for $k+1$. From Lemma 1, we obtain

$$\begin{aligned} P_{k+1}^{(u)}(n) &= \sum_{\substack{j=1 \\ (j,u)=1}}^{n} P_k^{(ju)}(n) = \sum_{\substack{j=1 \\ (j,u)=1}}^{n} \left(A_k f_k(ju) n^k + O(\theta(ju) n^{k-1} \log^{k-1} n)\right) \\ &= A_k f_k(u) n^k \sum_{\substack{j=1 \\ (j,u)=1}}^{n} f_k(j) + O\left(\theta(u) n^{k-1} \log^{k-1} n \sum_{j=1}^{n} \theta(j)\right). \end{aligned} \tag{8}$$

Here $\sum_{j=1}^{n} \theta(j) \le \sum_{j=1}^{n} \tau_2(j) = O(n \log n)$, where $\tau_2 = \tau$ is the divisor function.

Furthermore, by Lemma 2,

$$\sum_{\substack{j=1 \\ (j,u)=1}}^{n} f_k(j) = \sum_{\substack{de=j \le n \\ (j,u)=1}} \frac{\mu(d) k^{\omega(d)}}{\alpha_k(d)} = \sum_{\substack{d \le n \\ (d,u)=1}} \frac{\mu(d) k^{\omega(d)}}{\alpha_k(d)} \sum_{\substack{e \le n/d \\ (e,u)=1}} 1.$$

Using (7), we have

$$\begin{aligned} \sum_{\substack{j=1 \\ (j,u)=1}}^{n} f_k(j) &= \sum_{\substack{d \le n \\ (d,u)=1}} \frac{\mu(d) k^{\omega(d)}}{\alpha_k(d)} \left(\frac{n \phi(u)}{du} + O(\theta(u))\right) \\ &= \frac{\phi(u)}{u} n \sum_{\substack{d \le n \\ (d,u)=1}} \frac{\mu(d) k^{\omega(d)}}{d \alpha_k(d)} + O\left(\theta(u) \sum_{d \le n} \frac{k^{\omega(d)}}{d}\right), \end{aligned} \tag{9}$$

since $\alpha_k(d) > d$.

Hence, the main term of (9) is

$$\frac{\phi(u)}{u}n\sum_{\substack{d=1\\(d,u)=1}}^{\infty}\frac{\mu(d)k^{\omega(d)}}{d\alpha_k(d)}=\frac{\phi(u)}{u}n\prod_{p\nmid u}\left(1-\frac{k}{p(p+k-1)}\right)$$

$$=n\prod_{p}\left(1-\frac{k}{p(p+k-1)}\right)\prod_{p\mid u}\left(1-\frac{1}{p}\right)\left(1-\frac{k}{p(p+k-1)}\right)^{-1},$$

and its O-terms are

$$O\left(n\sum_{d>n}\frac{k^{\omega(d)}}{d^2}\right)=O\left(n\sum_{d>n}\frac{\tau_k(d)}{d^2}\right)=O(\log^{k-1}n)$$

by Lemma 3(b) and

$$O\left(\theta(u)\sum_{d\le n}\frac{k^{\omega(d)}}{d}\right)=O\left(\theta(u)\sum_{d\le n}\frac{\tau_k(d)}{d}\right)=O(\theta(u)\log^k n)$$

from Lemma 3(a).

Substituting into (8), we get

$$P_{k+1}^{(u)}(n)=A_k\prod_{p}\left(1-\frac{k}{p(p+k-1)}\right)f_k(u)\prod_{p\mid u}\left(1-\frac{1}{p}\right)\left(1-\frac{k}{p(p+k-1)}\right)^{-1}n^{k+1}$$

$$+O(n^k\log^{k-1}n)+O(\theta(u)n^k\log^k n)=A_{k+1}f_{k+1}(u)n^{k+1}+O(\theta(u)n^k\log^k n)$$

by an easy computation, which shows that the formula is true for $k+1$ and the proof is complete.

4. APPROXIMATION OF THE CONSTANTS A_k

Using the arithmetic mean-geometric mean inequality we have, for every $k\ge 2$ and every prime p,

$$\left(1-\frac{1}{p}\right)^{k-1}\left(1+\frac{k-1}{p}\right)<\frac{1}{k^k}\left((k-1)\left(1-\frac{1}{p}\right)+\left(1+\frac{k-1}{p}\right)\right)^k=1,$$

and obtain the series of positive terms,

$$\sum_{p}\log\left(\left(1-\frac{1}{p}\right)^{-k+1}\left(1+\frac{k-1}{p}\right)^{-1}\right)=\sum_{n=1}^{\infty}\log\left(\left(1-\frac{1}{p_n}\right)^{-k+1}\left(1+\frac{k-1}{p_n}\right)^{-1}\right)=-\log A_k, \tag{10}$$

where p_n denotes the n^{th} prime.

Furthermore, the Bernoulli-inequality yields

$$\left(1-\frac{1}{p}\right)^{k-1}\ge 1-\frac{k-1}{p},$$

hence,

$$\left(1-\frac{1}{p}\right)^{k-1}\left(1+\frac{k-1}{p}\right)\ge 1-\left(\frac{k-1}{p}\right)^2$$

for every $k\ge 2$ and every prime p.

Therefore, the N^{th}-order error R_N of series (10) can be evaluated as follows. Taking $N > k-1$, we have $p_N > k-1$ and

$$R_N = \sum_{n=N+1}^{\infty} \log\left(\left(1-\frac{1}{p_n}\right)^{-k+1}\left(1+\frac{k-1}{p_n}\right)^{-1}\right) \leq \sum_{n=N+1}^{\infty} \log\left(1-\left(\frac{k-1}{p_n}\right)^2\right)^{-1}$$

$$= \sum_{n=N+1}^{\infty} \log\left(1+\frac{(k-1)^2}{p_n^2-(k-1)^2}\right) < \sum_{n=N+1}^{\infty} \frac{(k-1)^2}{p_n^2-(k-1)^2}.$$

Now using that $p_n < 2n$, valid for $n \geq 5$, we have

$$R_N < \sum_{n=N+1}^{\infty} \frac{(k-1)^2}{4n^2-(k-1)^2} = \frac{k-1}{2}\sum_{n=N+1}^{\infty}\left(\frac{1}{2n-(k-1)}-\frac{1}{2n+(k-1)}\right)$$

$$= \frac{k-1}{2}\left(\frac{1}{2N-k+3}+\frac{1}{2N-k+5}+\cdots+\frac{1}{2N+k-1}\right) < \frac{(k-1)^2}{2(2N-k+3)}.$$

In order to obtain an approximation with r exact decimals, we use the condition

$$\frac{(k-1)^2}{2(2N-k+3)} \leq \frac{1}{2}\cdot 10^{-r}$$

and have $N \geq \frac{1}{2}((k-1)^2\cdot 10^r + k - 3)$. Consequently, for such an N,

$$A_k \approx \prod_{n=1}^{N}\left(1-\frac{1}{p_n}\right)^{k-1}\left(1+\frac{k-1}{p_n}\right)$$

with r exact decimals.

Choosing $r = 3$ and doing the computations on a computer (I used MAPLE V), we obtain the following approximate values of the numbers A_k:

$$A_2 = 0.607\ldots,\ A_3 = 0.286\ldots,\ A_4 = 0.114\ldots,\ A_5 = 0.040\ldots,$$
$$A_6 = 0.013\ldots,\ A_7 = 0.004\ldots,\ A_8 = 0.001\ldots.$$

Furthermore, taking into account that the factors of the infinite product giving A_k are less than 1, we obtain

$$A_{10} < \prod_{n=1}^{20}\left(1-\frac{1}{p_n}\right)^9\left(1+\frac{9}{p_n}\right) < 10^{-4},\quad A_{100} < \prod_{n=1}^{100}\left(1-\frac{1}{p_n}\right)^{99}\left(1+\frac{99}{p_n}\right) < 10^{-76}.$$

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