A COMBINATORIAL PROOF OF A RECURSIVE RELATION OF THE MOTZKIN SEQUENCE BY LATTICE PATHS

Wen-Jin Woan
Howard University, Washington, D.C. 20059
wwoan@fac.howard.edu

We consider those lattice paths in the Cartesian plane running from (0, 0) that use the steps from $S = \{U = (1,1)$ (an up-step), $L = (1,0)$ (a level-step), $D = (1,-1)$ (a down-step)$\}$. Let $A(n, k)$ be the set of all lattice paths ending at the point (n, k) and let $M(n)$ be the set of lattice paths in $A(n, 0)$ that never go below the x-axis. Let $a(n, k) = |A(n, k)|$ and $m_n = |M(n)|$, where m_n is called the *Motzkin number*. Here, we shall give a combinatorial proof of the three-term recursion of the Motzkin sequence,

$$(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2},$$

and also that

$$3 - \frac{6}{n+2} < \frac{m_n}{m_{n-1}} < 3 - \frac{4}{n+2}, \quad \lim_{n\to\infty} \frac{m_n}{m_{n-1}} = 3.$$

The first few Motzkin numbers are $m_0 = 1, 1, 2, 4, 9, 21, 51\ldots$. Let $B(n, k)$ denote the set of lattice paths in $A(n, k)$ that do not attain their highest value (i.e., maximum second coordinate) until the last step. Note that the last step of the paths in $B(n, k)$ is U. Let $b_{n,k} = |B(n, k)|$, then some entries of the matrices $(a_{n,k})$ and $(b_{n,k})$ are as follows:

$$\begin{bmatrix} n/k & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\ 0 & & & & & 1 & & & & \\ 1 & & & & 1 & 1 & 1 & & & \\ 2 & & & 1 & 2 & 3 & 2 & 1 & & \\ 3 & & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\ 4 & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \end{bmatrix},$$

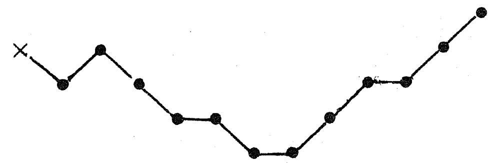
$$\begin{bmatrix} n/k & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 2 & 1 & 0 \\ 4 & 0 & 4 & 5 & 3 & 1 \end{bmatrix}.$$

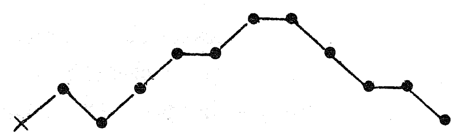
Lemma 1: There is a combinatorial proof for the equation $m_n = b_{n+1,1}$. See [1] and [3] for the cut and paste technique.

Proof: Let $P \in B(n+1, 1)$, remove the last step (U) and the reflection of the remaining is in $M(n)$. □

For example,

$$P = (DLDDUDLUULU)U \in B(12,1) \to DLDDUDLUULU$$
$$\to ULUUDULDDLD = Q \in M(11),$$

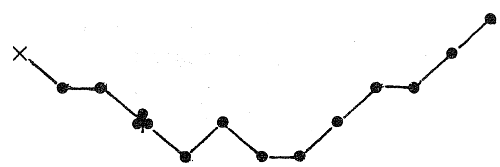
$P =$ 

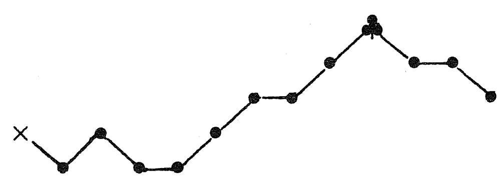
$Q =$ 

Theorem 2: There is a combinatorial proof for the equation $(n+1)b_{n+1,1} = a(n+1, 1)$. See also [5] for the proof and [1] and [3] for the cut and paste technique.

Proof: Let $S(n+1) = \{P^* : P \in B(n+1, 1),\ P^*$ with one marked vertex, which is one of the first $n+1$ vertices$\}$. Then $|S(n+1)| = (n+1)b_{n+1,1}$. Let $P^* \in S(n+1)$; this marked vertex partitions the path $P = FB$, where F is the front section and B is the back section. Then $Q = BF \in A(n+1, 1)$. Note that, graphically, the attached point is the leftmost highest point (the second coordinate) of Q. The converse starts with the leftmost highest point of Q in $A(n+1, 1)$ and reverse the above procedures. □

For example,

$\rightarrow P^* =$  $\in S(12)$,

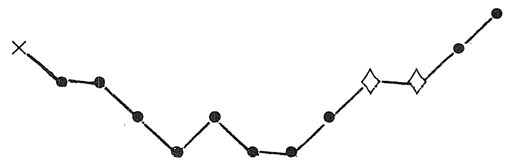
$\rightarrow Q =$  $\in A(12, 1)$.

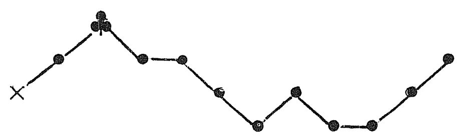
Proposition 3: The total number of L steps in $M(n)$ is the same as that in $B(n+1, 1)$ and is $a_{n,1}$.

Proof: From the proof of Lemma 1, the bijection between $M(n)$ and $B(n+1, 1)$ through reflection, it keeps the L steps. Hence, they have the same number of L steps.

Let $P = FLB \in B(n+1, 1)$ with L step. Then $Q = BF \in A(n, 1)$. Note that the joining point is the leftmost highest point in Q, since $P \in B(n+1, 1)$, by definition P reaches height 1 only at the end of the last step, the second coordinate of the L is less than or equal to 0; hence, any point in the subpath F from the initial point to L is lower or equal to the initial point and any point, before the terminal point, of the subpath B from L to the terminal point is of lower than the terminal point. This identification suggests the inverse mapping. □

For example,

$\rightarrow P = FLB =$  $\in B(12, 1)$,

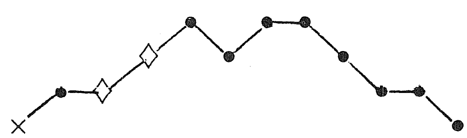
$Q = BF =$  $\in A(11, 1)$.

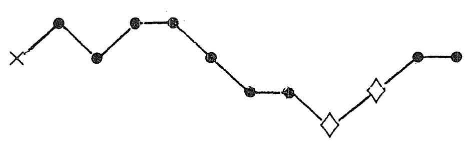
Proposition 4: There is a combinatorial proof for the equation

$$a_{n,0} = b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - a_{n,1}) = b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - nb_{n,1}).$$

Proof: Let $T(n) = \{P^e : P \in M(n),\ P^e \text{ is } P \text{ with an up-step marked}\}$. By Theorem 2 and Proposition 3, the number of level-steps among all paths in $M(n)$ is $a_{n,1} = nb_{n,1}$, and the total number of steps among all paths in $M(n)$ is $nm_n = nb_{n+1,1}$; hence, the total number of up-steps among all paths in $M(n)$ is $\frac{1}{2}(nb_{n+1,1} - nb_{n,1}) = |T(n)|$. Let $P^e = FUB \in T(n)$ with the U step marked, then $Q = BUF \in A(n, 0) - M(n)$ and the initial point of U in Q is the rightmost lowest point in Q. The inverse mapping starts with the rightmost lowest point. Note that $|M(n)| = m_n = b_{n+1,1}$. □

For example,

$P^e =$  $\in T(11)$,

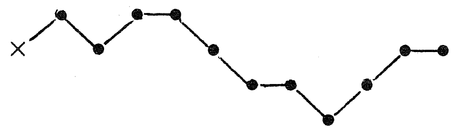
$Q =$  $\in A(11, 0)$.

Proposition 5: There is a combinatorial proof for the equation

$$a_{n,0} = a_{n-1,-1} + a_{n-1,0} + a_{n-1,1} = 2a_{n-1,1} + a_{n-1,0}$$
$$= 2(n-1)b_{n-1,1} + b_{n,1} + \frac{1}{2}((n-1)b_{n,1} - (n-1)b_{n-1,1}).$$

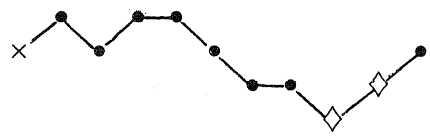
Proof: The first equality represents the partition of $A(n, 0)$ by the last step (U, L, or D), the second equality represents the symmetric property $a_{n-1,-1} = a_{n-1,1}$ and the last equality by Theorem 2 and Proposition 4. □

The following example shows the trail of one element for $n = 11$.



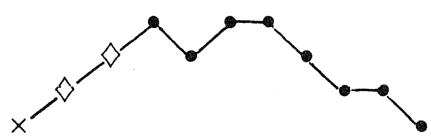
$\in A(11, 0)$.

Removing the last step, the second term of the first equality and the second term of the second equality,



$\in A(10, 0)$.

By Proposition 4, the second term of the third equality,



$\in T(10)$.

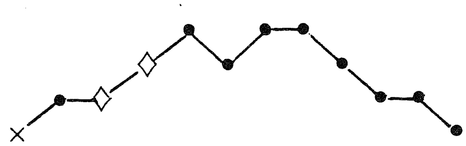
Theorem 6: There is a combinatorial proof for the equation

$$b_{n+1,1}+\frac{1}{2}(nb_{n+1,1}-nb_{n,1}) = 2(n-1)b_{n-1,1}+b_{n,1}+\left(\frac{1}{2}((n-1)b_{n,1}-(n-1)b_{n-1,1})\right).$$

Proof: The composition of the mappings in Proposition 4 and Proposition 5. □

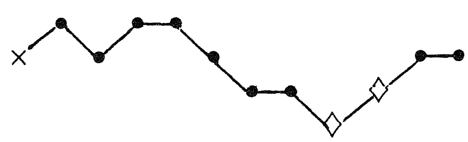
The following example shows the trail of one element for $n = 11$,

$P^e = FUB =$

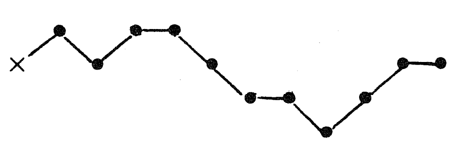


$\in T(11)$,

$Q = BUF =$

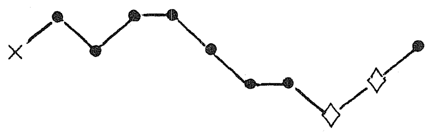


$\in A(11, 0)$,



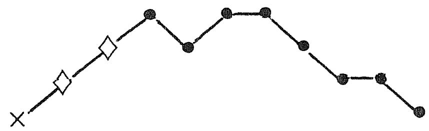
$\in A(11, 0)$.

Removing the last step,



$\in A(10, 0)$.

By Proposition 4,



$\in T(10)$.

The following result was proved in a combinatorial way in [2].

Theorem 7: $(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2}$.

Proof: By Theorem 6,

$$b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - nb_{n,1}) = 2(n-1)b_{n-1,1} + b_{n,1} + \left(\frac{1}{2}((n-1)b_{n,1} - (n-1)b_{n-1,1})\right).$$

By Lemma 1,

$$m_n + \frac{1}{2}(nm_n - nm_{n-1}) = 2(n-1)m_{n-2} + m_{n-1} + \left(\frac{1}{2}((n-2)m_{n-1} - (n-1)m_{n-2})\right).$$

Equivalently,

$$(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2}. \quad \square$$

Theorem 8: $3 - \frac{6}{n+2} < \frac{m_n}{m_{n-1}} < 3 - \frac{4}{n+2}$ for $n \geq 5$ and $\lim_{n\to\infty} \frac{m_n}{m_{n-1}} = 3$.

Proof: By Theorem 7, let

$$s_n := \frac{m_n}{m_{n-1}} = \frac{2n+1}{n+2} + \frac{3n-3}{n+2}\frac{m_{n-2}}{m_{n-1}} = \frac{2n+1}{n+2} + \frac{\frac{3n-3}{n+2}}{s_{n-1}},$$

$$a_n := \frac{2n+1}{n+2} = 2 - \frac{3}{n+2}, \quad b_n := \frac{3n-3}{n+2} = 3 - \frac{9}{n+2},$$

then

$$s_n = a_n + \frac{b_n}{s_{n-1}} \quad \text{and} \quad \frac{b_n}{s_n - a_n} = s_{n-1}.$$

If $s_{n-1} \leq 3$, then $\frac{b_n}{s_n - a_n} = s_{n-1} \leq 3$ and

$$s_n = a_n + \frac{b_n}{s_{n-1}} \geq 2 - \frac{3}{n+2} + \frac{3 - \frac{9}{n+2}}{3} = 3 - \frac{6}{n+2},$$

$$s_{n+1}=a_{n+1}+\frac{b_{n+1}}{s_n}\leq 2-\frac{3}{n+3}+\frac{3-\frac{9}{n+3}}{3-\frac{6}{n+2}}=2-\frac{3}{n+3}+\frac{\frac{3n}{n+3}}{\frac{3n}{n+2}}$$

$$=2-\frac{3}{n+3}+\frac{n+2}{n+3}=3-\frac{4}{n+3},$$

$$s_2=\frac{2}{1},\ s_3=\frac{4}{2},\ s_4=\frac{9}{4},\ s_5=\frac{21}{9},\ s_6=\frac{51}{21}<3.$$

By induction on both even and odd, we have the following:

$$3-\frac{6}{n+2}<\frac{m_n}{m_{n-1}}<3-\frac{4}{n+2},\quad \lim_{n\to\infty}\frac{m_n}{m_{n-1}}=3.\ \square$$

REFERENCES

1. N. Dershowitz & S. Zaks. "The Cycle Lemma and Some Applications." *European Journal of Combinatorics* **11** (1990):34-40.
2. S. Dulucq & J.-G. Penaud. "Interpretation Bijective D'une Recurrence des Nombres de Motzkin." Colloque Lacim 2000, http://www.lacim.uqam.ca/lacim2000/programme.html.
3. E. Pergola & R. Pinzani. "A Combinatorial Interpretation of the Area of Schroder Paths." *Electronic Journal of Combinatorics* **6** (1999).
4. R. A. Sulanke. "Moments of Generalized Motzkin Paths." *Journal of Integer Sequences* **3** (2000).
5. J. G. Wendel. "Left-Continuous Random Walk and the Lagrange Expansion." *Amer. Math. Monthly* **82** (1975):494-99.

AMS Classification Numbers: 05A10, 05A19, 11B65

❖❖❖