ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-583 ***Proposed by N. Gauthier, Royal Military College of Canada***

A Theorem on Generalized Fibonacci Convolutions

This is a generalization of Problem **B-858** by W. Lang (*The Fibonacci Quarterly* **36.3**, 1998).

Let $n \ge 0$, a, b be integers; also let A, B be arbitrary yet known real numbers and consider the generalized Fibonacci sequence $\{G_n \equiv A\alpha^n + B\beta^n\}_{n=-\infty}^{\infty}$, where

$$\alpha = \frac{1}{2}[1+\sqrt{5}],\ \beta = \frac{1}{2}[1-\sqrt{5}].$$

For m a nonnegative integer, prove the following generalized convolution theorem for the sequences $\{(a+n)^m\}_{n=-\infty}^{\infty}$ and $\{G_n\}_{n=-\infty}^{\infty}$,

$$\sum_{k=0}^{n}(a+k)^m G_{b-a-k} = \sum_{l=0}^{m} l![c_l^m(a)G_{b-a+l+1} - c_l^m(a+n+1)G_{b-a-n+1+l}],$$

where the set of coefficients $\{c_l^m(v);\ 0 \le m;\ 0 \le l \le m;\ v = a \text{ or } a+n+1\}$ satisfies the following second-order linear recurrence relation

$$c_l^{m+1}(v) = (v+l)c_l^m(v) + c_{l-1}^m(v);\ c_{l=0}^{m=0}(v) = 1,\ c_{l=0}^{m=1}(v) = v,\ c_{l=1}^{m=1}(v) = 1$$

with the understanding that $c_{-1}^m(v) \equiv 0$ and that $c_{m+1}^m(v) \equiv 0$.

Prob. **B-858** follows as a special case if one sets $a = 0$, $m = 1$, $b = n$, and $A = -B = (\alpha-\beta)^{-1}$ in the above theorem. Indeed, one then gets that

$$G_n = F_n,\ c_0^1(0) = 0,\ c_1^1(0) = 1,\ c_0^1(n+1) = n+1, \text{ and } c_1^1(n+1) = 1$$

and the result follows directly.

H-584 ***Proposed by Paul S. Bruckman, Sacramento, CA***

Prove the following identity:

$$(F_{n+4} + L_{n+3})^5 + (F_n + L_{n+1})^5 + (2F_{n+1} + L_{n+2})^5$$
$$= (2F_{n+3} + L_{n+2})^5 + (F_{n+2})^5 + (5F_{n+2})^5 + 1920F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}.$$

SOLUTIONS

Some Operator!

H-571 ***Proposed by D. Tsedenbayar, Mongolian Pedagogical University, Warsaw, Poland (Vol. 39, no. 1, February 2001)***

Prove: If $(T_\alpha f)(t) = t^\alpha \int_0^t f(s)\,ds$, with $\alpha \in \mathbf{R}$, then

$$(T_\alpha^n f)(t) = \frac{t^\alpha}{(\alpha+1)^{(n-1)}(n-1)!}\int_0^t (t^{\alpha+1} - s^{\alpha+1})^{n-1} f(s)\,ds, \quad \text{for } \alpha \neq -1$$

and

$$(T_\alpha^n f)(t) = \frac{1}{t(n-1)!}\int_0^t \left(\ln\frac{t}{s}\right)^{n-1} f(s)\,ds, \quad \text{for } \alpha = -1.$$

Remark: If $\alpha = -1$, then T_{-1} is a Cesaro operator; if $\alpha = 0$, then T_0 is a Volterra operator.

Solution by Paul S. Bruckman, Sacramento, CA

Our proof is by induction on n. We let $S(\alpha)$ denote the set of positive integers n such that the statements of the problem are true. Note that the statements of the problem are true for $n = 1$, since they reduce to the definitions of $(T_\alpha)(f(t))$. That is, $1 \in S(\alpha)$.

Suppose $n \in S(\alpha)$. Then $(T_\alpha)^{n+1}(f(t)) = (T_\alpha)(T_\alpha)^n(f(t))$.

If $\alpha \neq -1$,

$$\begin{aligned}(T_\alpha)^{n+1}(f(t)) &= (T_\alpha)\frac{t^\alpha}{(\alpha+1)^{n-1}(n-1)!}\int_0^t (t^{\alpha+1} - s^{\alpha+1})^{n-1} f(s)\,ds \\ &= \frac{t^\alpha}{(\alpha+1)^{n-1}(n-1)!}\int_0^t s^\alpha \int_0^s (s^{\alpha+1} - u^{\alpha+1})^{n-1} f(u)\,du\,ds \\ &= \frac{t^\alpha}{(\alpha+1)^{n-1}(n-1)!}\int_0^t f(u) \int_u^t s^\alpha (s^{\alpha+1} - u^{\alpha+1})^{n-1}\,ds\,du \\ &= \frac{t^\alpha}{(\alpha+1)^{n-1}(n-1)!}\int_0^t f(u) \left.\frac{(s^{\alpha+1} - u^{\alpha+1})^n}{(\alpha+1)n}\right]_u^t du \\ &= \frac{t^\alpha}{(\alpha+1)^n(n)!}\int_0^t f(s)(t^{\alpha+1} - s^{\alpha+1})^n\,ds,\end{aligned}$$

which is the statement of the first part of the problem ($\alpha \neq -1$) for $n+1$. That is, $n \in S(\alpha) \Rightarrow (n+1) \in S(\alpha)$ if $\alpha \neq -1$.

The second part of the problem (for $\alpha = -1$) is treated similarly. In this case,

$$\begin{aligned}(T_{-1})^{n+1}(f(t)) &= (T_{-1})(1/t(n-1)!)\int_0^t (\log t/s)^{n-1} f(s)\,ds \\ &= (1/t(n-1)!)\int_0^t 1/s \int_0^s (\log s/u)^{n-1} f(u)\,du\,ds \\ &= (1/t(n-1)!)\int_0^t f(u) \int_u^t 1/s(\log s/u)^{n-1}\,ds\,du\end{aligned}$$

$$= (1/t(n-1)!)\int_0^t f(u)/n\left[(\log s/u)^n\right]_u^t du$$

$$= (1/t(n)!)\int_0^t f(u)(\log t/u)^n du$$

$$= (1/t(n)!)\int_0^t f(s)(\log t/s)^n ds.$$

This is the statement of the problem for $\alpha = -1$, $n+1$. Therefore, $n \in S(-1) \Rightarrow (n+1) \in S(-1)$. We have shown that, for all real α, $n \in S(\alpha) \Rightarrow (n+1) \in S(\alpha)$. The desired results follow by induction.

Sum Problem

H-572 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 39, no. 2, May 2001)

Prove the following, where $\varphi = \alpha^{-1}$:

$$\sum_{n=0}^{\infty}\{\varphi^{5n+1}/(5n+1)+\varphi^{5n+3}/(5n+2)-\varphi^{5n+4}/(5n+3)-\varphi^{5n+4}/(5n+4)\} = (\pi/25)(50-10\sqrt{5})^{1/2}.$$

Solution by Kenneth B. Davenport, Frackville, PA

Since, for $|x|<1$,

$$\frac{1}{1-x^5} = 1+x^5+x^{10}+x^{15}+\cdots = \sum_{n=0}^{\infty} x^{5n}, \tag{1}$$

we let, for $-1<x<1$,

$$A(x) = \int_0^{\varphi}\frac{1}{1-x^5}dx = \sum_{n=0}^{\infty}\frac{\varphi^{5n+1}}{(5n+1)}, \tag{2}$$

$$B(x) = \varphi\int_0^{\varphi}\frac{x}{1-x^5}dx = \sum_{n=0}^{\infty}\frac{\varphi^{5n+3}}{(5n+2)}, \tag{3}$$

$$C(x) = -\varphi\int_0^{\varphi}\frac{x^2}{1-x^5}dx = -\sum_{n=0}^{\infty}\frac{\varphi^{5n+4}}{(5n+3)}, \tag{4}$$

$$D(x) = -\int_0^{\varphi}\frac{x^3}{1-x^5}dx = \sum_{n=0}^{\infty}\frac{\varphi^{5n+4}}{(5n+4)}. \tag{5}$$

Making use of an integral expression:

$$\int\frac{x^m}{1-x^n}dx = -\frac{1}{n}\cos\frac{2(m+1)\pi}{n}\log\left(1-2x\cos\frac{2\pi}{n}+x^2\right)-\frac{1}{n}\cos\frac{4(m+1)\pi}{n}\log\left(1-2x\cos\frac{4\pi}{n}+x^2\right)$$
$$-\frac{1}{n}\cos\frac{6(m+1)\pi}{n}\log\left(1-2x\cos\frac{6\pi}{n}+x^2\right)-\cdots+\frac{2}{n}\sin\frac{2(m+1)\pi}{n}\arctan\frac{x\sin\frac{2\pi}{n}}{1-x\cos\frac{2\pi}{n}}$$
$$+\frac{2}{n}\sin\frac{4(m+1)\pi}{n}\arctan\frac{x\sin\frac{4\pi}{n}}{1-x\cos\frac{4\pi}{n}}+\frac{2}{n}\sin\frac{6(m+1)\pi}{n}\arctan\frac{x\sin\frac{6\pi}{n}}{1-x\cos\frac{6\pi}{n}}+\cdots-\frac{1}{n}\log(1-x).$$

From *Tables of Indefinite Integrals* by G. Petit Bois (Dover Publications, 1961), we derive the following.

For $A(x)$:

$$
\begin{aligned}
&-\frac{1}{5}\cos\frac{2\pi}{5}\log\left(1-2x\cos\frac{2\pi}{5}+x^2\right)\\
&-\frac{1}{5}\cos\frac{4\pi}{5}\log\left(1-2x\cos\frac{4\pi}{5}+x^2\right)\\
&+\frac{2}{5}\sin\frac{2\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{2\pi}{5}}{1-x\cos\frac{2\pi}{5}}\right]\\
&+\frac{2}{5}\sin\frac{4\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{4\pi}{5}}{1-x\cos\frac{4\pi}{5}}\right].
\end{aligned}
\tag{6}
$$

For $B(x)$:

$$
\begin{aligned}
&-\frac{1}{5}\cos\frac{4\pi}{5}\log\left(1-2x\cos\frac{2\pi}{5}+x^2\right)\\
&-\frac{1}{5}\cos\frac{8\pi}{5}\log\left(1-2x\cos\frac{4\pi}{5}+x^2\right)\\
&+\frac{2}{5}\sin\frac{4\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{2\pi}{5}}{1-x\cos\frac{2\pi}{5}}\right]\\
&+\frac{2}{5}\sin\frac{8\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{4\pi}{5}}{1-x\cos\frac{4\pi}{5}}\right].
\end{aligned}
\tag{7}
$$

For $C(x)$:

$$
\begin{aligned}
&+\frac{1}{5}\cos\frac{6\pi}{5}\log\left(1-2x\cos\frac{2\pi}{5}+x^2\right)\\
&+\frac{1}{5}\cos\frac{12\pi}{5}\log\left(1-2x\cos\frac{4\pi}{5}+x^2\right)\\
&-\frac{2}{5}\sin\frac{6\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{2\pi}{5}}{1-x\cos\frac{2\pi}{5}}\right]\\
&-\frac{2}{5}\sin\frac{12\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{4\pi}{5}}{1-x\cos\frac{4\pi}{5}}\right].
\end{aligned}
\tag{8}
$$

For $D(x)$:

$$
\begin{aligned}
&+\frac{1}{5}\cos\frac{8\pi}{5}\log\left(1-2x\cos\frac{2\pi}{5}+x^2\right)\\
&+\frac{1}{5}\cos\frac{16\pi}{5}\log\left(1-2x\cos\frac{4\pi}{5}+x^2\right)\\
&-\frac{2}{5}\sin\frac{8\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{2\pi}{5}}{1-x\cos\frac{2\pi}{5}}\right]\\
&-\frac{2}{5}\sin\frac{16\pi}{5}\tan^{-1}\left[\frac{x\sin\frac{4\pi}{5}}{1-x\cos\frac{4\pi}{5}}\right].
\end{aligned}
\tag{9}
$$

And now, keeping in mind that (7) is multiplied by the factor φ, (8) by $-\varphi$, and (9) by -1, we observe that (6) and (9) when summed cancel the *logarithmic* parts due to sign and likewise (7) and (8) when summed will cancel the *logarithmic* parts. Thus, upon evaluating (6) and (9) as well as (7) and (8) between the bounds 0 and φ, one will then have:

(6) + (9) =

$$+\frac{2}{5}\sin\frac{2\pi}{5}\cdot\frac{\pi}{5}+\frac{2}{5}\sin\frac{4\pi}{5}\tan^{-1}\left[\frac{\varphi\sin\frac{4\pi}{5}}{1-\varphi\cos\frac{4\pi}{5}}\right]$$

$$-\frac{2}{5}\sin\frac{8\pi}{5}\cdot\frac{\pi}{5}-\frac{2}{5}\sin\frac{16\pi}{5}\tan^{-1}\left[\frac{\varphi\sin\frac{4\pi}{5}}{1-\varphi\cos\frac{4\pi}{5}}\right]; \tag{10}$$

(7) + (8) =

$$+\frac{2}{5}\sin\frac{4\pi}{5}\cdot\frac{\pi}{5}+\frac{2}{5}\sin\frac{8\pi}{5}\tan^{-1}\left[\frac{\varphi\sin\frac{4\pi}{5}}{1-\varphi\cos\frac{4\pi}{5}}\right]$$

$$-\frac{2}{5}\sin\frac{6\pi}{5}\cdot\frac{\pi}{5}-\frac{2}{5}\sin\frac{12\pi}{5}\tan^{-1}\left[\frac{\varphi\sin\frac{4\pi}{5}}{1-\varphi\cos\frac{4\pi}{5}}\right]. \tag{11}$$

And now, noting that

$$\left[\sin\frac{4\pi}{5}-\sin\frac{16\pi}{5}+\varphi\sin\frac{8\pi}{5}-\varphi\sin\frac{12\pi}{5}\right]=0,$$

we may simplify (10) and (11) to obtain

$$\frac{2\pi}{25}\left[\sin\frac{2\pi}{5}-\sin\frac{8\pi}{5}+\sin\frac{4\pi}{5}-\sin\frac{6\pi}{5}\right]. \tag{12}$$

Analytically, this reduces to the expression:

$$\frac{\pi}{25}(10+2\sqrt{5})^{1/2}+\left(\frac{3-\sqrt{5}}{2}\right)^{1/2}(10+2\sqrt{5})^{1/2}=\frac{\pi}{25}(10+2\sqrt{5})^{1/2}+(20-8\sqrt{5})^{1/2}. \tag{13}$$

And (13) is equivalent to

$$\frac{\pi}{25}(50-10\sqrt{5})^{1/2}.$$

Also solved by F. Ovidiu, H.-J. Seiffert, and the proposer.

Fee Fi Fo Fum

H-573 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 39, no. 2, May 2001)***

"By definition, a magic matrix is a square matrix whose lines, columns, and two main diagonals all add up to the same sum. Consider a 3×3 magic matrix Φ whose elements are the following combinations of the n^{th} and $(n+1)^{\text{th}}$ Fibonacci numbers:

$$\Phi_{11}=3F_{n+1}+F_n;\quad \Phi_{12}=F_{n+1};\quad \Phi_{13}=2F_{n+1}+2F_n;$$
$$\Phi_{21}=F_{n+1}+2F_n;\quad \Phi_{22}=2F_{n+1}+F_n;\quad \Phi_{23}=3F_{n+1};$$
$$\Phi_{31}=2F_{n+1};\quad \Phi_{32}=3F_{n+1}+2F_n;\quad \Phi_{33}=F_{n+1}+F_n.$$

Find a closed-form expression for Φ^m, where m is a *positive* integer, and determine all the values of m for which it too is a magic matrix."

Solution by the proposer

It is well known that the elements of a 3×3 magic matrix can generally be written in the form:

$$\begin{array}{lll} \Phi_{11} = a+b; & \Phi_{12} = a-(b+c); & \Phi_{13} = a+c; \\ \Phi_{21} = a-(b-c); & \Phi_{22} = a; & \Phi_{23} = a+(b-c); \\ \Phi_{31} = a-c; & \Phi_{32} = a+(b+c); & \Phi_{33} = a-b. \end{array}$$

In the present situation, $a = F_{n+3}$, $b = F_{n+1}$, $c = F_n$.

Now define three magic matrices, as follows:

$$A \equiv \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; \quad B \equiv \frac{1}{\sqrt{3}}\begin{pmatrix} +1 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -1 \end{pmatrix}; \quad C \equiv \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 & +1 \\ +1 & 0 & -1 \\ -1 & +1 & 0 \end{pmatrix}.$$

Then $\Phi = \alpha A + \beta B + \gamma C$, where $\alpha \equiv 3F_{n+3}$, $B \equiv \sqrt{3}F_{n+1}$, $\gamma \equiv \sqrt{3}F_n$.

Next, for m an integer, one can simply verify the following multiplication properties:

$$A^m = A, m > 0; \;\; AB = BA = AC = CA = N; \;\; BC = -CB;$$

$$B^2 = -C^2 = I - A; \;\; B^{2m} = B^2, m > 0; \;\; B^{2m+1} = B, m \ge 0.$$

N and I are the 3×3 null and identity matrices, respectively. Consequently,

$$\Phi^2 = \alpha^2 A + (\beta^2 - \gamma^2)B^2,$$

and since A, B commute, with $AB = BA = N$, and

$$\beta^2 - \gamma^2 = (\beta - \gamma)(\beta + \gamma) = 3F_{n+2}F_{n-1},$$

we find that

$$\begin{aligned} \Phi^{2m} &= [\alpha^2 A + (\beta^2 - \gamma^2)B^2]^m = \sum_{k=0}^{m}\binom{m}{k}(\alpha^2 A)^k[(\beta^2-\gamma^2)B^2]^{m-k} \\ &= \alpha^{2m}A + (\beta^2 - \gamma^2)^m B^2 \\ &= 3^m[3^m F_{n+3}^{2m} - F_{n+2}^m F_{n-1}^m]A + [3^m F_{n+2}^m F_{n-1}^m]I \end{aligned}$$

for m a ***positive*** integer. Furthermore, for m a ***nonnegative*** integer,

$$\begin{aligned} \Phi^{2m+1} &= (\alpha A + \beta B + \gamma C)[\alpha^{2m}A + [(\beta^2-\gamma^2)^m B^2] \\ &= \alpha^{2m+1}A + (\beta^2 - \gamma^2)^m[\beta B + \gamma C] \\ &= 3^{2m+1}F_{n+3}^{2m+1}A + 3^{m+1/2}F_{n-1}^m F_{n+2}^m[F_{n+1}B + F_n C]. \end{aligned}$$

Odd powers of the magic matrix Φ are always magic as well, whereas even powers are only so if $\beta^2 = \gamma^2$. This completes the solution.

Also solved by P. Bruckman.

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