THE BRAHMAGUPTA POLYNOMIALS IN TWO COMPLEX VARIABLES AND THEIR CONJUGATES

R. Rangarajan

Department of Mathematics, Mysore University, Manasagangotri, 570 006, India
E-mail: rajra63@yahoo.com

H. S. Sudheer

Department of Mathematics, Kasturbha College, Shimoga, Karnataka, India

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1. INTRODUCTION

The Brahmagupta matrix and polynomials in two real variables were first introduced by Suryanarayan [7]. Later they were extended to two complex variables [8]. There is yet another way to extend naturally from the real variables case to the complex variables case. This is done by using two complex variables with their conjugates. In this paper we will explore this way of generalizing the matrix and the polynomials. This method yields quite different results than the ones developed in [8].

We define the Brahmagupta matrix, see (1) below, involving two complex variables as well as their conjugates and show that it generates a class of homogeneous polynomials. The two complex variables z and w lie in two distinct complex planes. This space is denoted $\mathbb{C}\times\mathbb{C}$ or $\mathbb{C}^2$. A typical member of this space has the form $\zeta=(z,w)$. Following [8], the points in $\mathbb{C}^2$ can be identified naturally with the points of $\mathbb{R}^4$ by the scheme:

$$(z,w)\in\mathbb{C}^2\leftrightarrow(x+iy,u+iv)\leftrightarrow(x,y,u,v)\in\mathbb{R}^4.$$

The polynomials generated by the matrix contain some of the well-known real polynomials like Chebychev polynomials of the first and second kind and Morgan-Voyce polynomials, among others. Thus, the paper provides a unified approach to the study of Brahmagupta polynomials.

In this paper we study the Brahmagupta matrix and the Brahmagupta polynomials in two complex variables and their conjugates. This study is similar to those in [7] and [8] and provides a natural way to extend them from the real case to the complex case. The emerging polynomials have a unique feature, namely, their real and imaginary parts form only two polynomials instead of four, involving essentially two variables. However, they have to be studied in two different cases depending on the nature of the variables: (i) both real; (ii) one real and the other purely imaginary. It is interesting to note that in the former case the Brahmagupta matrix and Brahmagupta polynomials are particular cases of those given in [7]; in the latter case, they are special cases of those given in [8]. In fact, Section 2 is clearly different from [7] and [8]. Section 5 is intended to show that the extended class of polynomials contain many of the well-known polynomials.

2. BRAHMAGUPTA MATRIX WITH COMPLEX ENTRIES

Let $z=x+iy$ and $w=u+iv$ be two complex variables and let $\bar{z}=x-iy$ and $\bar{w}=u-iv$ be their conjugates. Let $t\neq 0$ be a fixed real number. Consider the matrix

$$B_J=B_J(z,w)=\begin{bmatrix} z & w \\ t\bar{w} & \bar{z}\end{bmatrix}=B(x,u)+JB(y,v), \tag{1}$$

where

$$B(\xi, \eta) = \begin{bmatrix} \xi & \eta \\ t\eta & \xi \end{bmatrix} \text{ and } J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Let $\beta = \det(B_J) = |z|^2 - t|w|^2$. It is clear that, if $y = 0$ and $v = 0$, then (1) reduces to the real case [7]. Let $\mathbf{B_J}$ denote the set of all matrices of the form B_J. Define $\overline{B}_J = B_J(\overline{z}, \overline{w})$. $\mathbf{B_J}$ shows that, if $B_k = B_J(z_k, w_k)$, then $\mathbf{B_J}$ satisfies the following properties:

$$B_1B_2 \neq B_2B_1,\ \overline{B}_1\overline{B}_2 = \overline{B}_1\overline{B}_2,\ B_J\overline{B}_J \neq \overline{B}_J B_J.$$

Thus, if the entries of B_J are real, then $\mathbf{B_J}$ forms a commutative subgroup of $GL(2, R)$. But in the present case, $\mathbf{B_J}$ is a *noncommutative* subgroup of $GL(2, C)$.

Let $\beta = \det(B_J) \neq 0$. Set $\alpha^2 = x^2 - \beta$. Notice that α is real if $x^2 - \beta > 0$ and α is imaginary if $x^2 - \beta < 0$. The eigenvalues of B_J are $\lambda_{\pm} = x \pm \alpha$, with corresponding eigenvectors $E_{\pm} = [\pm w, \alpha \mp iy]^T$, where T denotes the transpose. Using the eigenrelations, B_J can be diagonalized in the form

$$\begin{bmatrix} z & w \\ t\overline{w} & \overline{z} \end{bmatrix} = \frac{1}{2w\alpha}\begin{bmatrix} w & -w \\ \alpha - iy & \alpha + iy \end{bmatrix}\begin{bmatrix} x+\alpha & 0 \\ 0 & x-\alpha \end{bmatrix}\begin{bmatrix} \alpha + iy & w \\ -\alpha + iy & w \end{bmatrix}. \tag{2}$$

Define

$$\begin{bmatrix} z & w \\ t\overline{w} & \overline{z} \end{bmatrix}^n = \begin{bmatrix} z_n & w_n \\ t\overline{w}_n & \overline{z}_n \end{bmatrix}.$$

Then, using the above eigenrelations, we find that

$$\begin{bmatrix} z & w \\ t\overline{w} & \overline{z} \end{bmatrix}^n = \frac{1}{2w\alpha}\begin{bmatrix} w & -w \\ \alpha - iy & \alpha + iy \end{bmatrix}\begin{bmatrix} (x+\alpha)^n & 0 \\ 0 & (x-\alpha)^n \end{bmatrix}\begin{bmatrix} \alpha + iy & w \\ -\alpha + iy & w \end{bmatrix}.$$

From the above result, we derive the following Binet forms for z_n and w_n:

$$z_n = \frac{1}{2\alpha}[(\alpha + iy)(x+\alpha)^n + (\alpha - iy)(x-\alpha)^n], \tag{3}$$

$$w_n = \frac{w}{2\alpha}[(x+\alpha)^n - (x-\alpha)^n]. \tag{4}$$

Let us consider the two cases: (a) α is real; (b) α is imaginary.

Case (a). For α real, we can separate the real and imaginary parts of $z_n = x_n + iy_n$ and $w_n = u_n + iv_n$ and obtain

$$\begin{aligned} &(i) && x_n = \frac{1}{2}[(x+\alpha)^n + (x-\alpha)^n], \\ &(ii) && y_n = \frac{y}{2\alpha}[(x+\alpha)^n - (x-\alpha)^n], \\ &(iii) && u_n = \frac{u}{2\alpha}[(x+\alpha)^n - (x-\alpha)^n], \\ &(iv) && v_n = \frac{v}{2\alpha}[(x+\alpha)^n - (x-\alpha)^n]. \end{aligned} \tag{5}$$

Set

$$\frac{\alpha_n}{\alpha} = \frac{y_n}{y} = \frac{u_n}{u} = \frac{v_n}{v}. \tag{6}$$

From the above results, we see that instead of the four forms x_n, y_n, u_n, and v_n, there are essentially two forms to consider, namely,

$$x_n = \frac{1}{2}[(x+\alpha)^n + (x-\alpha)^n] \quad \text{and} \quad \alpha_n = \frac{1}{2}[(x+\alpha)^n - (x-\alpha)^n]. \tag{7}$$

We can generate x_n and α_n by the matrix

$$A = A(x, \alpha) = \begin{bmatrix} x & \alpha \\ \alpha & x \end{bmatrix}.$$

Case (b). For α imaginary, let us write $\alpha = i\hat{\alpha}$. Following a similar procedure as for the real case, we find that

$$\frac{\hat{\alpha}_n}{\alpha} = \frac{\hat{y}_n}{y} = \frac{\hat{u}_n}{u} = \frac{\hat{v}_n}{v}, \tag{8}$$

where $\hat{y}_n$ is obtained by replacing α by $i\alpha$ in (5, ii) and similarly we define $\hat{x}_n$, $\hat{u}_n$, $\hat{v}_n$, and $\hat{\alpha}_n$. In relation (7), replacing α by $i\hat{\alpha}$, we find that

$$\hat{x}_n = \frac{1}{2}[(x+i\hat{\alpha})^n + (x-i\hat{\alpha})^n] \quad \text{and} \quad i\hat{\alpha}_n = \frac{1}{2}[(x+i\hat{\alpha})^n - (x-i\hat{\alpha})^n]. \tag{9}$$

From (7) and (9), we see that $x_n \pm \alpha_n = (x \pm \alpha)^n$ and $\hat{x}_n \pm i\hat{\alpha}_n = (x \pm i\hat{\alpha})^n$. Similarly, we can generate $\hat{x}_n$ and $\hat{\alpha}_n$ by the matrix

$$\hat{A} = \hat{A}(x, i\hat{\alpha}) = \begin{bmatrix} x & i\hat{\alpha} \\ i\hat{\alpha} & x \end{bmatrix}.$$

3. PROPERTIES OF A AND $\hat{A}$

Notice that the determinant of A as well as that of $\hat{A}$ is $x^2 - \alpha^2 \neq 0$. Since

$$A(x_1, \alpha_1)A(x_2, \alpha_2) = A(x_2, \alpha_2)A(x_1, \alpha_1),$$

the set of matrices of the form A commute. Set

$$A_n = A^n = \begin{bmatrix} x & \alpha \\ \alpha & x \end{bmatrix}^n = \begin{bmatrix} x_n & \alpha_n \\ \alpha_n & x_n \end{bmatrix}.$$

The Binet forms of A are given by (7). x_n and α_n satisfy the following recurrence relations:

$$x_{n+1} = xx_n + \alpha\alpha_n; \quad \alpha_{n+1} = x\alpha_n + \alpha x_n. \tag{10}$$

From the recurrence relation (10), we derive the *three-term recurrence relations* satisfied by x_n and α_n:

$$x_{n+1} = 2xx_n - (x^2 - \alpha^2)x_{n-1}; \quad \alpha_{n+1} = 2x\alpha_n - (x^2 - \alpha^2)\alpha_{n-1}.$$

It is clear that, if α is imaginary, the three-term recurrence relation becomes

$$\hat{x}_{n+1} = 2x\hat{x}_n - (x^2 + \hat{\alpha}^2)\hat{x}_{n-1}; \quad \hat{\alpha}_{n+1} = 2x\hat{\alpha}_n - (\hat{x}^2 + \hat{\alpha}^2)\hat{\alpha}_{n-1}.$$

If $\xi_n = x_n + \alpha_n$ and $\eta_n = x_n - \alpha_n$, then $\xi^n = \xi_n$ and $\eta^n = \eta_n$.

From the above results we see that, for real α,

$$e^A = e^x \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix}.$$

To show this, we write $2x_k = \xi_k + \eta_k$ and $2\alpha_k = \xi_k - \eta_k$. Since

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad \text{and} \quad A_k = \begin{bmatrix} x_k & \alpha_k \\ \alpha_k & x_k \end{bmatrix},$$

we express x_k and α_k in terms of ξ and η and obtain the desired results. Notice that $\det e^A = e^{2x}$.

On the other hand, if α is imaginary, we replace α by $i\hat{\alpha}$ and follow a similar reasoning to show that

$$e^{\hat{A}} = e^x \begin{bmatrix} \cos\hat{\alpha} & i\sin\hat{\alpha} \\ i\sin\hat{\alpha} & \cos\hat{\alpha} \end{bmatrix}.$$

In this case also, $\det e^{\hat{A}} = e^{2x}$.

x_n and α_n can be extended to the negative integers also by defining $x_{-n} = x_n\beta^{-n}$ and $\alpha_{-n} = -\alpha_n\beta^{-n}$. Then we will have

$$A^{-n} = \begin{bmatrix} x & \alpha \\ \alpha & x \end{bmatrix}^{-n} = \begin{bmatrix} x_{-n} & \alpha_{-n} \\ \alpha_{-n} & x_{-n} \end{bmatrix} = A_{-n};$$

here we have used the property

$$\left(\begin{bmatrix} x & \alpha \\ \alpha & x \end{bmatrix}^{-1}\right)^n = \left(\frac{1}{\beta}\begin{bmatrix} x & -\alpha \\ -\alpha & x \end{bmatrix}\right)^n = \frac{1}{\beta^n}\begin{bmatrix} x_n & -\alpha_n \\ -\alpha_n & x_n \end{bmatrix}.$$

Notice that $A^0 = I$, the identity matrix. A similar result holds for $\hat{A}^{-n}$.

4. RECURRENCE RELATIONS

From the Binet forms (7) and (9), the reader may verify the following.

Recurrence Relations:

$$\begin{aligned}
&(i) && x_{m+n} = x_m x_n \pm \alpha_m \alpha_n, \\
&(ii) && \alpha_{m+n} = x_m \alpha_m + \alpha_m x_n, \\
&(iii) && \beta^{2n} x_{m-n} = x_m x_n \mp \alpha_m \alpha_n, \\
&(iv) && \beta^{2n} \alpha_{m-n} = x_n \alpha_m \mp x_m \alpha_n, \\
&(v) && x_{m+n} + \beta^{2n} x_{m-n} = 2x_m x_n, \\
&(vi) && \alpha_{m+n} + \beta^{2n} \alpha_{m-n} = 2x_n \alpha_m, \\
&(vii) && x_{m+n} - \beta^{2n} x_{m-n} = \pm 2\alpha_m \alpha_n, \\
&(viii) && \alpha_{m+n} - \beta^{2n} \alpha_{m-n} = 2x_m \alpha_n,
\end{aligned} \tag{11}$$

where the top sign is chosen if α is real; if α is imaginary, the bottom sign is chosen. Notice that (v) and (vi) are the generalizations of the three-term recurrence relations.

Let $\sum_{k=1}^{n} = \sum$. Again using the Binet forms, the reader may verify the following

$$
\begin{aligned}
&(i) && \sum x_k = \frac{\beta^2 x_n - x_{n+1} + x - \beta^2}{\beta^2 - 2x + 1},\\
&(ii) && \sum \alpha_k = \frac{\beta^2 \alpha_n - \alpha_{n+1} + \alpha}{\beta^2 - 2x + 1},\\
&(iii) && \sum x_k^2 = \frac{\beta^2 x_{2n} - x_{2n+2} + x_2 - \beta^2}{2(\beta^2 - 2x_2 + 1)} + \frac{\beta^2(\beta^{2n} - 2)}{2(\beta^2 - 1)},\\
&(iv) && \sum \alpha_k^2 = \frac{\beta^2 \alpha_{2n} - \alpha_{2n+2} + \alpha_2 - \beta^2}{2(\beta^2 - 2\alpha_2 + 1)} + \frac{\beta^2(\beta^{2n} - 1)^2}{2(\beta^2 - 1)},\\
&(v) && 2\sum x_k x_{n+1-k} = n x_{n+1} + \frac{\beta^2 \alpha_n}{\alpha},\\
&(vi) && 2\sum \alpha_k \alpha_{n+1-k} = n x_{n+1} - \frac{\beta^2 \alpha_n}{\alpha},\\
&(vii) && 2\sum x_k \alpha_{n-k+1} = 2\sum \alpha_k x_{n-k+1} = n\alpha_{n+1}.
\end{aligned}
\tag{12}
$$

(12, v, vi, vii) are convolution formulas. For α imaginary, a set of similar formulas holds.

From the Binet forms for (7) we see that, for $\alpha > 0$, x_n and α_n satisfy

The Limiting Properties:

$$\lim_{n\to\infty} \frac{x_n}{\alpha_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n}{x_{n-1}} = \lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n-1}} = x + \alpha.$$

The Divisors of x_{2n} and α_{2n}:

From (11 i) we see that, if α is imaginary, then $x + i\alpha$ and $x - i\alpha$ are factors of x_{2n} for real α.

From (11 ii) we see that x_n and α_n are factors of α_{2n}. The last statement can be generalized: If r divides s, then x_r and α_r are factors of α_s.

5. BRAHMAGUPTA POLYNOMIALS

With the help of the binomial expansions for $x_n \pm \alpha_n = (x \pm \alpha)^n$, we find that

$$
\begin{aligned}
x_n &= x^n + \binom{n}{2} x^{n-2}\alpha^2 + \binom{n}{4} x^{n-4}\alpha^4 + \cdots,\\
\alpha_n &= n x^{n-1}\alpha + \binom{n}{3} x^{n-3}\alpha^3 + \binom{n}{5} x^{n-5}\alpha^5 + \cdots.
\end{aligned}
$$

Similarly, expanding $\hat{x}_n \pm i\hat{\alpha}_n = (x \pm i\hat{\alpha})^n$, we obtain

$$
\begin{aligned}
\hat{x}_n &= x^n - \binom{n}{2} x^{n-2}\hat{\alpha}^2 + \binom{n}{4} x^{n-4}\hat{\alpha}^4 - + \cdots,\\
\hat{\alpha}_n &= n x^{n-1}\hat{\alpha} - \binom{n}{3} x^{n-3}\hat{\alpha}^3 + \binom{n}{5} x^{n-5}\hat{\alpha}^5 - + \cdots.
\end{aligned}
$$

For α real, the first few polynomials of x_n and α_n are:

$$
\begin{gathered}
x_0 = 1,\ x_1 = x,\ x_2 = x^2 + \alpha^2,\ x_3 = x^3 + 3x\alpha^2,\ x_4 = x^4 + 6x^2\alpha^2 + y^4, \ldots;\\
\alpha_0 = 0,\ \alpha_1 = \alpha, \alpha_2 = 2x\alpha,\ \alpha_3 = 3x^2\alpha + \alpha^3,\ \alpha_4 = 4x\alpha^3 + 4x^3\alpha, \ldots.
\end{gathered}
$$

Similarly, for α imaginary, we can write the first few polynomials of $\hat{x}_n$ and $\hat{\alpha}_n$.

Special Cases of the Polynomials:

a. Brahmagupta sequences

If $x=2$ and $\alpha=\sqrt{3}$, the Binet forms reduce to

$$2x_n=(2+\sqrt{3})^n+(2-\sqrt{3})^n;\ 2\sqrt{3}u_n=(2+\sqrt{3})^n-(2-\sqrt{3})^n.$$

These sequences appear in obtaining Heron triangles with consecutive integer sides [1]; $2x_n$ denotes the middle side and $2u_n$ denotes the height of the triangle.

b. Lucas and Fibonacci sequences L_n and F_n

In B_J in (1), set $x=y=u=\frac{1}{2}$, $v=0$, and $t=6$. Then we get $\beta^2=\alpha^2-x^2=1$, $\alpha=\frac{\sqrt{5}}{2}$, and

$$2x_n=\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n=L_n,\ 2\sqrt{5}u_n=\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n=F_n,$$

and, in this case, we have

$$2B_J(L_n+iF_n,F_n)=\begin{bmatrix}L_n+iF_n & F_n\\ 6F_n & L_n-iF_n\end{bmatrix}.$$

c. Pell sequences

In B_J, if we set $x=y=u=1$, $v=0$, and $t=3$, we get $\beta=-1$, $\alpha=\sqrt{2}$, and x_n and u_n reduce to Pell sequences given by:

$$2x_n=(1+\sqrt{2})^n+(1-\sqrt{2})^n,\ 2\sqrt{2}u_n=(1+\sqrt{2})^n-(1-\sqrt{2})^n.$$

Also, B_n becomes

$$B_J(x_n+iy_n,y_n)=\begin{bmatrix}x_n+iy_n & y_n\\ 3y_n & x_n-iy_n\end{bmatrix}.$$

d. Brahmagupta polynomials

If $v=0=y$, then x_n and y_n reduce to the Brahmagupta polynomials in the real case:

$$x_n=\frac{1}{2}[(x+y\sqrt{t})^n+(x-y\sqrt{t})^n],\ y_n=\frac{1}{2\sqrt{t}}[(x+y\sqrt{t})^n-(x-y\sqrt{t})^n].$$

The properties of these polynomials have been studied in [7].

e. The Chebyshev polynomials

Set $\beta=1$, $\alpha=\sqrt{x^2-1}$, and $u=1$, then

$$x_n=\frac{1}{2}[(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n]=T_n(x),$$
$$u_n=\frac{u}{2\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^n-(x-\sqrt{x^2-1})^n]=U_n(x).$$

The Chebyshev polynomials occur in many branches of mathematics like Interpolation Theory, Orthogonal Polynomials, Approximation Theory, Numerical Analysis, etc. [6].

f. Polynomials similar to Chebyshev polynomials

If we set $\beta=-1$ and $\alpha=\sqrt{x^2+1}$, we obtain polynomials similar to the Chebyshev polynomials:

$$x_n = \frac{1}{2}[(x+\sqrt{x^2+1})^n + (x-\sqrt{x^2-1})^n] = V_n(x),$$
$$\frac{\alpha_n}{\alpha} = \frac{1}{2\sqrt{x^2+1}}[(x+\sqrt{x^2+1})^n + (x-\sqrt{x^2-1})^n] = W_n(x).$$

g. Morgan-Voyce polynomials

If x is replaced by $(x+2)/2$ and α by $\sqrt{x^2+4x}/2$ in the matrix A, then the det $A = 1$. If, in addition, $u = 1$, then

$$2x_n = \left(\frac{x+2+\sqrt{x^2+4x}}{2}\right)^n + \left(\frac{x+2-\sqrt{x^2+4x}}{2}\right)^n,$$
$$2\frac{u_n}{u}\sqrt{x^2+4x} = \left(\frac{x+2+\sqrt{x^2+4x}}{2}\right)^n - \left(\frac{x+2-\sqrt{x^2+4x}}{2}\right)^n = B_n,$$

where B_n is the Morgan-Voyce polynomial [4], [9]. The three-term recurrence relation for these polynomials are $B_n = (2+x)B_{n-1} - B_{n-2}$. Morgan-Voyce polynomials are used in the analysis of ladder networks and electric line theory [4], [9].

h. Catalan numbers

If $x = 1$ and $\alpha^2 = 1+4u$ in (5), we find that

$$2x_n = (1+\sqrt{1+4u})^n + (1-\sqrt{1+4u})^n,$$
$$2u_n = \frac{1}{\sqrt{1+4u}}[(1+\sqrt{1+4u})^n - (1-\sqrt{1+4u})^n].$$

Both x_n and u_n appear in the study of Catalan numbers [2].

Let α be real. Then we find, from (7),

$$\frac{\partial x_n}{\partial x} = \frac{\partial \alpha_n}{\partial \alpha} = n x_{n-1}, \quad \frac{\partial x_n}{\partial \alpha} = \frac{\partial \alpha_n}{\partial x} = n\alpha_{n-1}.$$

From the above relations, we infer that x_n and α_n are the polynomial solutions of the wave equation:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \alpha^2}\right) X = 0.$$

On the other hand, let α be imaginary. Put $\alpha = i\hat{\alpha}$. Then we find, from (9),

$$\begin{aligned} \frac{\partial \hat{x}_n}{\partial x} &= \frac{\partial \hat{\alpha}_n}{\partial \hat{\alpha}} = n\hat{x}_{n-1}, \\ \frac{\partial \hat{x}_n}{\partial \hat{\alpha}} &= \frac{-\partial \hat{\alpha}_n}{\partial x} = -n\hat{\alpha}_{n-1}. \end{aligned} \tag{13}$$

From these relations, we infer that $\hat{x}_n$ and $\hat{\alpha}_n$ are the polynomial solutions of the Laplace equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \hat{\alpha}^2}\right) X = 0.$$

6. GENERATING FUNCTIONS

We shall now show that the generating functions for z_n and w_n are:

$$(i)\ \sum_{n=0}^{\infty} z_n s^n = \frac{1-\bar{z}s}{1-2xs+\beta^2 s^2};\quad (ii)\ \sum_{n=0}^{\infty} w_n s^n = \frac{ws}{1-2xs+\beta^2 s^2}. \tag{14}$$

We shall assume that s is real; then we can separate the real and imaginary parts on both sides to obtain the following generating functions for x_n and α_n:

$$(i)\ \sum_{n=0}^{\infty} x_n s^n = \frac{1-xs}{1-2xs+\beta s^2};\quad (ii)\ \sum_{n=0}^{\infty} \alpha_n s^n = \frac{\alpha s}{1-2xs+\beta s^2}. \tag{15}$$

To show (13), we use the standard result: For $\|B_J s\| < 1$, we have

$$\sum_{n=0}^{\infty} (B_J s)^n = (I - B_J s)^{-1}.$$

Now,

$$I - B_J s = \begin{bmatrix} 1-zs & -ws \\ -t\bar{w}s & 1-\bar{z}s \end{bmatrix},$$

$$\det(I - B_J s) = 1-(z+\bar{z})s + (|z|^2 - t|w|^2)s^2 = 1-2xs+\beta s^2,$$

and

$$(1-2xs+\beta s^2)\sum_{n=0}^{\infty}(B_J s)^n = \begin{bmatrix} 1-\bar{z}s & ws \\ t\bar{w}s & 1-zs \end{bmatrix}.$$

The claim (14) follows from the above result.

It is known that, if $F(s)$ and $L(s)$ are generating functions of F_n and L_n, respectively, then $F(s) = e^{L(s)}$ [3]. This result can be generalized to the generating functions of x_n and α_n. Let

$$X(s) = \sum_{k=1}^{\infty} x_k s^k,\quad A(s) = \sum_{k=1}^{\infty} \frac{\alpha_k}{\alpha} s^{k-1},\quad \chi(s) = \sum_{k=1}^{\infty} x_k \frac{s^k}{k}.$$

Notice that $s\chi'(s) = X(s)$. Now, we state this result as the following theorem.

Theorem: $e^{2\chi(s)} = A(s)$.

Proof: Set $\xi = x+\alpha$ and $\eta = x-\alpha$. Then

$$\xi+\eta = 2x,\quad \xi\eta = x^2-\alpha^2 = \beta,\quad 2x_n = (\xi^n+\eta^n),\quad 2\alpha_n = (\xi^n - \eta^n).$$

Now consider

$$\begin{aligned} \chi(s) &= \sum_{k=1}^{\infty} x_k \frac{s^k}{k} \\ &= \frac{1}{2}\left[\xi s + \xi^2\frac{s^2}{2} + \cdots + \xi^n \frac{s^n}{n} + \cdots\right] + \frac{1}{2}\left[\eta s + \eta^2 \frac{s^2}{2} + \cdots + \eta^n \frac{s^n}{n} + \cdots\right] \\ &= -\frac{1}{2}[\ln(1-\xi s) + \ln(1-\eta s)] = -\frac{1}{2}\ln(1-2xs+\beta s^2), \end{aligned}$$

which implies $2\chi(s) = \ln A(s)$ or $e^{2\chi(s)} = A(s)$.

All the infinite series summation properties involving reciprocals of x_n and y_n developed in [7] can be extended to x_n and α_n (or $\hat{\alpha}_n$). Since the arithmetic goes through without any changes, we do not list them here.

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