THE LEAST INTEGER HAVING p FIBONACCI REPRESENTATIONS, p PRIME

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1. INTRODUCTION

Given a positive integer N, a representation of N as a sum of distinct Fibonacci numbers in descending order is a Fibonacci representation of N. Let $R(N)$ be the number of Fibonacci representations of N. For example, $R(58) = 7$, since 58 can be written as:

$$\begin{array}{lll} 55+3 & 34+21+3 & 34+13+8+3 \\ 55+2+1 & 34+21+2+1 & 34+13+8+2+1 \\ & & 34+13+5+3+2+1 \end{array}$$

Any positive integer N can be represented uniquely as the sum of distinct, nonconsecutive Fibonacci numbers; this representation is the Zeckendorf representation of N, denoted Zeck N. In particular, Zeck $58 = 55+3 = F_{10}+F_4$, in subscript notation.

The subscripts of the Fibonacci numbers appearing in Zeck N allow calculation of $R(N)$ by using reduction formulas [3], [4]. If Zeck $N = F_{n+k}+K$, where $K = F_n + \cdots + F_t < F_{n+1}$, then

$$R(N) = R(F_{n+2q}+K) = qR(K)+R(F_{n+1}-K-2), \quad k = 2q, \tag{1.1}$$

$$R(N) = R(F_{n+2q+1}+K) = (q+1)R(K), \quad k = 2q+1. \tag{1.2}$$

Further, subscripts in Zeck N can be shifted downward c to calculate $R(N-1)$,

$$R(N-1) = R(F_{n+k-c}+F_{n-c}+\cdots+F_{t-c}-1), \quad t \ge c+2. \tag{1.3}$$

Lastly, tables for $R(N)$ contain palindromic lists. For N within successive intervals $F_n \le N \le F_{n+1}-2$, the values for $R(N)$ satisfy the symmetric property

$$R(F_{n+1}-2-M) = R(F_n+M), \; 0 \le M \le F_{n-1}, \; n \ge 3. \tag{1.4}$$

The table for $R(N)$ repeats patterns within intervals and subintervals although with increasingly larger values; indeed, $R(N)$ appears fractal in nature. What interests us, however, is the inverse problem: Given a value n, write an integer N such that $R(N) = n$ or, most interesting of all, find the least N having exactly n representations as sums of distinct Fibonacci numbers.

Let A_n be the least positive integer having exactly n Fibonacci representations. Then $\{A_n\} = \{1, 3, 8, 16, 24, 37, 58, 63, \ldots\}$, but while the first 330 values for A_n are listed in [6], A_n is given by formula only for special values of n. However, when p is prime, all Fibonacci numbers used in Zeck A_p have even subscripts. The sequence $\{B_n\}$ of the next section arises from an attempt to make sense of $\{A_n\}$ when $n = p$ is prime.

2. EVEN-ZECK INTEGERS AND THE BOUNDING SEQUENCE $\{B_n\}$

If an integer N has a prime number of Fibonacci representations, then the subscripts of the Fibonacci numbers appearing in Zeck N have the same parity. Since $R(F_{2k+1}) = R(F_{2k})$, we concentrate upon even subscripts. We will call a positive integer whose Zeckendorf representation contains only even-subscripted Fibonacci numbers an *even-Zeck integer*.

Here we study a bounding sequence $\{B_n\}$, where $B_n \geq A_n$, $n \geq 1$. We let B_n be the least even-Zeck integer having exactly n Fibonacci representations. Note that $A_n = B_n$ whenever A_n is an even-Zeck integer.

We begin by listing even-Zeck N and computing $R(N)$ for N in our restricted domain. In Table 2.1, we underline the first occurrence of each value for $R(N)$ and list subscripts only for Zeck N. Notice that 2^k integers N have Zeck N beginning with $F_{2(k+1)}$. For N in the interval $F_{2k} \leq N \leq F_{2k+1} - 2$, $R(N)$ takes on values in a palindromic list which begins with $k = R(F_{2k})$ and ends with $k = R(F_{2k+1} - 2)$, with central value 2. Interestingly, every third entry for $R(N)$ is even.

TABLE 2.1. $R(N)$ for Even-Zeck N, $1 \leq N \leq 88$

$R(N)$	N	Zeck N	$R(N)$	N	Zeck N
$\underline{1}$	$\underline{1}$	2	5	55	10
$\underline{2}$	$\underline{3}$	4	4	56	10,2
1	4	4,2	$\underline{7}$	$\underline{58}$	10,4
$\underline{3}$	$\underline{8}$	6	3	59	10,4,2
2	9	6,2	$\underline{8}$	$\underline{63}$	10,6
3	11	6,4	5	64	10,6,2
1	12	6,4,2	7	66	10,6,4
$\underline{4}$	$\underline{21}$	8	2	67	10,6,4,2
3	22	8,2	7	76	10,8
$\underline{5}$	$\underline{24}$	8,4	5	77	10,8,2
2	25	8,4,2	8	79	10,8,4
5	29	8,6	3	80	10,8,4,2
3	30	8,6,2	7	84	10,8,6
4	32	8,6,4	4	85	10,8,6,2
1	33	8,6,4,2	5	87	10,8,6,4
			1	88	10,8,6,4,2

In Table 2.1, the listed values for $R(N)$ for $N = F_{10} + K$ can be obtained by writing the values (1), 4, 3, 5, 2, ..., from $R(N)$ for $N = F_8 + K$, interspersed with their sums: (1), $\underline{5}$, 4, $\underline{7}$, 3, $\underline{8}$, 5, $\underline{7}$, 2, ..., the first half of the palindromic sequence of $R(N)$ values for $N = F_{10} + K$, where, of course, the second half repeats. The first (1) arises from $R(F_t - 1) = 1$, $t \geq 1$; the algorithm computes $R(N)$ for even-Zeck N in the interval $F_{2k} \leq N \leq F_{2k+1} - 1$, using values obtained from the preceding interval for N.

Theorem 2.1: If N is an even-Zeck integer such that Zeck N ends in F_{2c}, $c \geq 2$, $F_{2k} \leq N \leq F_{2k+1} - 1$, and N^* is the even-Zeck integer preceding N, then

$$R(N) = R(N+1) + R(N^*). \tag{2.1}$$

Further, $R(N+1) = R(M)$ and $R(N^*) = R(M^*)$, where M^* is the even-Zeck integer preceding M in the interval $F_{2k-2} \leq M \leq F_{2k-1} - 1$.

Proof: We will use (1.3) to shift subscripts in computing $R(N+1)$ and $R(N^*)$. If $N = F_{2k} + \cdots + F_{2c+2p} + F_{2c}$, $c \geq 2$, then the even-Zeck integer preceding N is

$$\begin{aligned} N^* &= F_{2k} + \cdots + F_{2c+2p} + (F_{2c-2} + \cdots + F_4 + F_2) \\ &= F_{2k} + \cdots + F_{2n+2p} + (F_{2c-1} - 1) \\ &= N - F_{2c} + F_{2c-1} - 1 = N - F_{2c-2} - 1. \end{aligned} \tag{2.2}$$

While $(N-1)$ is not an even-Zeck integer, we can apply (1.3) to shift each subscript down $(2c-2)$ to obtain an even-Zeck integer,

$$\begin{aligned} R(N-1) &= R(F_{2k}+\cdots+F_{2c+2p}+F_{2c}-1) = R(F_{2k-2c+2}+\cdots+F_{2c+2p-2c+2}+F_{2c-2c+2}-1) \\ &= R(F_{2k-2c+2}+\cdots+F_{2p+2}+F_2-1) = R(F_{2k-2c+2}+\cdots+F_{2p+2}) = R(K), \end{aligned} \tag{2.3}$$

where K is an even-Zeck integer. Similarly, shifting subscripts down $2c-2$ in (2.2), we obtain $R(N^*)=R(N-1)$. From [3], $R(N)=R(N+1)+R(N-1)$ for any integer N such that Zeck N ends in F_{2c}, $c\geq 2$. The rest of Theorem 2.1 follows from similar subscript reductions, so that

$$R(N+1)=R(F_{2k-2}+\cdots+F_{2c+2p-2}+F_{2c-2})=R(M), \tag{2.4}$$

and $R(N^*)=R(F_{2k-2}+\cdots+F_{2c+2p-2}+F_{2c-2}-F_{2c-4}-1)=R(M^*)$. □

When we list the 2^k values for $R(N)$ for even-Zeck N in the interval $F_{2k}\leq N\leq F_{2k+1}-1$, the corresponding values for N can be found by numbering the entries for $R(N)$. For example, in Table 2.1, 66 is the 7^{th} entry in the interval $F_{10}\leq N\leq F_{11}-1$ (the 6^{th} entry after 55), and $6=2^2+2^1$ corresponds to $F_{2(2+1)}+F_{2(1+1)}$; Zeck $66=F_{10}+F_6+F_4$. If $R(N)$ is the m^{th} entry in the interval $F_{2k}\leq N\leq F_{2k+1}-1$, and if $(m-1)=2^p+\cdots+2^w$, then the associated even-Zeck integer N has Zeck $N=F_{2k}+F_{2(p+1)}+\cdots+F_{2(w+1)}$. Further, the list is palindromic; the m^{th} entry for $R(N)$ equals the $(2^{k-1}-m)^{\text{th}}$ entry.

Since A_p is an even-Zeck integer when p is prime, $B_p=A_p$ for prime p, and $B_n\geq A_n$ for all $n\geq 1$. The first occurrences of $R(N)$ in Table 2.1 give us $\{B_n\}=\{1,3,8,21,24,\_,58,63,\dots\}$, where B_6 is as yet unknown. Table 2.2 lists the first 89 values for $\{B_n\}$, from computation of $R(N)$ for even-Zeck N, $1\leq N<F_{23}$.

TABLE 2.2. B_n for $1\leq n\leq 89$

n	B_n	n	B_n	n	B_n	n	B_n
1*	1	23*	1011	45	3134	67*	7166
2*	3	24	1063	46*	2990	68	7221
3*	8	25	1053	47*	2752	69*	7200
4	21	26	1045	48	6975	70	8158
5*	24	27*	1066	49*	2985	71*	7310
6	144	28	2608	50*	3019	72	18719
7*	58	29*	1050	51	6930	73*	7831
8*	63	30	1139	52	6917	74*	8187
9	147	31*	1160	53*	6967	75	7954
10	155	32	2650	54	19298	76*	7205
11*	152	33	2642	55*	3024	77	18295
12	173	34*	1155	56	7163	78	18164
13*	168	35	2663	57	6972	79*	7815
14	385	36	2807	58	7297	80	7959
15	398	37*	2647	59*	7349	81*	7925
16	461	38	6841	60	6933	82	18918
17*	406	39	2969	61*	7218	83*	18154
18*	401	40	2749	62	7836	84	18240
19*	435	41*	2736	63	7171	85	18112
20	1215	42	7145	64	7315	86	19083
21*	440	43*	2757	65	7208	87	18167
22	1016	44	2791	66	7899	88	18146
						89*	7920

In Table 2.2, * denotes $B_n = A_n$, which is always true for $n = F_k$ or L_k, and when n is prime. However, while we can have $B_n = A_n$ when n is composite, the most irregularly occurring values for B_n are when n is even.

Theorem 2.2: The following special values for n have $A_n = B_n$:

$$n = F_{k+1} \quad B_n = F_{2k} + F_{2k-4} + F_{2k-8} + F_{2k-12} + \cdots, \quad k \geq 2; \tag{2.5}$$

$$n = L_{k-1} \quad B_n = F_{2k} + F_{2k-6} + F_{2k-10} + F_{2k-14} + \cdots, \quad k \geq 3. \tag{2.6}$$

Proof: A_n has the above values for the given values of n from [1]. Since in these two cases A_n is an even-Zeck integer, $A_n = B_n$. □

From computation of the first 610 values for B_n, it appears that if Zeck n begins with F_k, that is, $F_k < n < F_{k+1}$, then Zeck B_n begins with F_{2k}, F_{2k+2}, or F_{2k+4}; this has not been proved. However, F_{m+1} is the largest value for $R(M)$ in the interval $F_{2m} \leq M \leq F_{2m+1}$, and all other values for $R(M)$ which appear in that interval have Zeck n beginning with F_m or a smaller Fibonacci number. Note that we are relating n and B_n in an interesting way, since the subscripts in Zeck N are used to compute $R(N)$.

3. PROPERTIES OF $\{B_n\}$

Theorem 3.1: If N is an even-Zeck integer such that $F_{2k} \leq N < F_{2k+1}$, and if $M = F_{k+1}^2 - 1$, then the three largest values occurring for $R(N)$ are:

$$\begin{array}{lll} R(N) = n & N = B_n & \\ F_{k+1} & M = F_{k+1}^2 - 1, & k \geq 2; \quad (3.1) \\ F_{k+1} - F_{k-4} & M + 5(-1)^k, & k \geq 6; \quad (3.2) \\ F_{k+1} - F_{k-4} - F_{k-8} & M + 39(-1)^k, & k \geq 9. \quad (3.3) \end{array}$$

For even-Zeck N in this interval, the following values for $R(N)$ do not occur:

$$R(N) = F_{k+1} - p, \; 1 \leq p \leq F_{k-4} + F_{k-8} - 1, \; k \geq 9, \tag{3.4}$$

except for $p = F_{k-4}$. In particular,

$$R(N) = F_{k+1} - 1, \; k \geq 7,$$

is a missing value.

Proof: From [1], M is the smallest integer having F_{k+1} Fibonacci representations; Zeck M appears in (2.5). Tables for $R(N)$ show palindromic behavior within each interval for N as well as "peaks" containing clusters of values where $N = B_n$. The "peak value" is the sum of two adjacent values for $R(M)$ at the "peak" of the preceding interval $F_{2k-2} \leq M < F_{2k-1}$ from the formation of the table for $R(N)$.

Table 3.1 exhibits behavior near the primary peak value $R(N) = F_{k+1}$ for the interval

$$F_{2k} + F_{2k-4} \leq N < F_{2k} + F_{2k-3}.$$

Recalling (2.1), when Zeck N ends in $F_{2c} \geq F_4$, $R(N) = R(N+1) + R(N^*)$, where N^* is the even-Zeck integer preceding N. Since we are looking at consecutive even-Zeck N in Table 3.1, the formula for each value of $R(N)$ can be proved by induction, $k \geq 6$.

TABLE 3.1. $R(N)$ for Even-Zeck N, $F_{2k}+F_{2k-4}\le N<F_{2k}+F_{2k-3}$

k odd: $M=F_{k+1}^2-1=F_{2k}+F_{2k-4}+\cdots F_{14}+F_{10}+F_6$

	$R(N)$	N	Zeck N ends with:
	...	...	...
	F_k+F_{k-5}	$M-8$	$...+F_{14}+F_{10}$
	L_{k-2}	$M-7$	$...+F_{14}+F_{10}+F_2$
$N=B_n$	$F_{k+1}-F_{k-4}$	$M-5$	$...+F_{14}+F_{10}+F_4$
	F_{k-1}	$M-4$	$...+F_{14}+F_{10}+F_4+F_2$
$N=B_n$	F_{k+1}	M	$...+F_{14}+F_{10}+F_6$
	F_k	$M+1$	$...+F_{14}+F_{10}+F_6+F_2$
	L_{k-1}	$M+3$	$...+F_{14}+F_{10}+F_6+F_4$
	F_{k-2}	$M+4$	$...+F_{14}+F_{10}+F_6+F_4+F_2$
	$F_{k+1}-L_{k-4}$	$M+13$	$...+F_{14}+F_{10}+F_8$
	...	...	...

k even: $M=F_{k+1}^2-1=F_{2k}+F_{2k-4}+\cdots+F_{12}+F_8+F_4$

	$R(N)$	N	Zeck N ends with:
	...	...	...
	$F_{k+1}-L_{k-4}$	$M-13$	$...+F_{12}+F_6+F_4$
	F_{k-2}	$M-12$	$...+F_{12}+F_6+F_4+F_2$
	L_{k-1}	$M-3$	$...+F_{12}+F_8$
	F_k	$M-2$	$...+F_{12}+F_8+F_2$
$N=B_n$	F_{k+1}	M	$...+F_{12}+F_8+F_4$
	F_{k-1}	$M+1$	$...+F_{12}+F_8+F_4+F_2$
$N=B_n$	$F_{k+1}-F_{k-4}$	$M+5$	$...+F_{12}+F_8+F_6$
	L_{k-2}	$M+6$	$...+F_{12}+F_8+F_6+F_2$
	F_k+F_{k-5}	$M+8$	$...+F_{12}+F_8+F_6+F_4$
	F_{k-3}	$M+9$	$...+F_{12}+F_8+F_6+F_4+F_2$
	...	...	...

We show that $R(N)=B_n$ for $n=F_{k+1}-F_{k-4}$ because we cannot get the same result for a smaller N. In Table 3.1, N is in the interval $F_{2k}+F_{2k-4}<N<F_{2k}+F_{2k-3}$. To have $R(N)=F_{k+1}-F_{k-4}$ for a smaller N, we must have $F_{2k}<N<F_{2k}+F_{2k-4}$. From (2.6), L_{k-1} is the largest value for $R(N)$ for even-Zeck N in the interval $F_{2k}+F_{2k-6}<N<F_{2k}+F_{2k-4}$, where $L_{k-1}=F_k+F_{k-2}<F_{k+1}-F_{k-4}=F_k+F_{k-2}+F_{k-5}$, so $R(N)=F_{k+1}-F_{k-4}$ cannot occur for $N<F_{2k}+F_{2k-4}$, establishing (3.2). Equation (3.3) follows in a similar manner. □

Corollary 3.1.1: For $n=F_{k+1}-F_{k-4}$ as in Theorem 3.1, $A_n=B_n$ for $k\ge 7$.

When N is any positive integer, $R(N)$ displays "peak" values near $R(N)=F_{k+1}$ similar to those listed in Table 3.1 for even-Zeck integers N. The three largest values for $R(N)$, when N is any positive integer, $F_{2k}\le N<F_{2k+1}$, are F_{k+1}, $F_{k+1}-F_{k-5}=4F_{k-2}$, and $F_{k+1}-F_{k-4}$. When $n=4F_{k-2}$, $A_n=M+8(-1)^{k+1}$ for $M=F_{k+1}^2-1$. The values for $R(N)=F_{k+1}-p$, $1\le p\le F_{k-5}-1$, $k\ge 6$, are missing for N in that interval.

A similar "secondary peak" in the lists for $R(N)$ clusters around L_{k-1}, both for N any positive integer and for N an even-Zeck integer; hence, Theorem 3.2.

Theorem 3.2: If $M = F_{2k} + F_{2k-6} + F_{2k-10} + \cdots = F_{2k} + F_{k-2}^2 - 1$, then when

$$n = L_{k-1} \qquad B_n = M, \qquad k \geq 5; \tag{3.5}$$

$$n = L_{k-1} - L_{k-6} \qquad B_n = M + 5(-1)^{k-1}, \quad k \geq 7; \tag{3.6}$$

$$n = L_{k-1} - L_{k-6} - L_{k-10} \qquad B_n = M + 39(-1)^{k-1}, \quad k \geq 11. \tag{3.7}$$

Corollary 3.2.1: For $n = L_{k-1} - L_{k-6}$ as in Theorem 3.2, $A_n = B_n$ for $k \geq 9$.

4. UNANSWERED QUESTIONS

Theorem 3.1 shows some values for $R(N)$ that are missing within each interval for even-Zeck N, $F_{2k} \leq N < F_{2k+1}$, $k \geq 9$. In what interval will those "missing values" first appear? The value $n = R(N)$ always occurs for some even-Zeck N, since, in the worst case scenario, $n = R(F_{2n})$. But when is $\{B_n\}$ complete?

Conjecture 3.1.3: If $R(N)$ is calculated for all even-Zeck N, $N < F_{2k+5}$, then $\{B_n\}$ is complete for $1 \leq n \leq F_k$. If $F_k < n < F_{k+1}$, then $F_{2k} < B_n < F_{2k+5}$.

Finding the least integer having p Fibonacci representations, p prime, is an unsolved problem.

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