CONVOLUTIONS FOR JACOBSTHAL-TYPE POLYNOMIALS

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1. PRELIMINARIES

Object of the Paper

Basically, the purpose of this paper is to present data on convolution polynomials $J_n^{(k)}(x)$ and $j_n^{(k)}(x)$ for Jacobsthal and Jacobsthal-Lucas polynomials $J_n(x)$ and $j_n(x)$, respectively, and, more specifically, on the corresponding convolution numbers arising when $x=1$.

Our information will roughly parallel and, therefore, should be compared with that offered for Pell and Pell-Lucas polynomials $P_n(x)$ and $Q_n(x)$, respectively, in [7] and [8] in particular.

Properties of $J_n(x)$ and $j_n(x)$ may be found in [5] and [6, p. 138]. Originally $J_n(x)$ was investigated by the Norwegian mathematician Jacobsthal [9]. For ease of reference, it is thought desirable to reproduce a few essential features of $J_n(x)$ and $j_n(x)$ in the next subsection.

Background articles of relevance on convolutions which could be consulted with benefit are [1], [2], and [3]. But observe that in [3] the x has to be replaced by $2x$ for our $J_n(x)$.

Convolution Arrays

Convolution numbers, symbolized by $J_n^{(k)}(1) \equiv J_n^{(k)}$ and $j_n^{(k)}(1) \equiv j_n^{(k)}$, where k represents the "order" of the convolution and n the sequence index, may be displayed in a *convolution array* (pattern). When $k=0$, the ordinary Jacobsthal numbers $J_n^{(0)} \equiv J_n$ and the Jacobsthal-Lucas numbers $j_n^{(0)} \equiv j_n$ are generated.

Readers of [3, p. 401] will be aware that the n^{th}-order convolution sequence for $J_n^{(k)}$ appears there as columns of a matrix. As the convolution array for $j_n^{(k)}$ does not seem to have been previously recorded, we shall disclose its details in Table 2.

Mathematical Background

Definitions

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x), \quad J_0(x)=0, \; J_1(x)=1. \tag{1.1}$$

$$j_{n+2}(x) = j_{n+1}(x) + 2xj_n(x), \quad j_0(x)=2, \; j_1(x)=1. \tag{1.2}$$

For $0 \le n \le 10$, $J_n(x)$ and $j_n(x)$ are recorded in [6] in Tables 1 and 2, respectively, to which the reader is encouraged to refer.

Special Cases

$x=1$: Jacobsthal numbers $J_n(1)=J_n$ and Jacobsthal-Lucas numbers $j_n(1)=j_n$.

$x=\frac{1}{2}$: $J_n(\frac{1}{2})=F_n$, $j_n(\frac{1}{2})=L_n$ (the n^{th} Fibonacci and Lucas numbers).

It follows that Tables 1 and 2 in [6] with (1.1) and (1.2) thus generate the number sequences

$$\{J_n(1)\} = 0, 1, 1, 3, 5, 11, 21, 43, \ldots, \tag{1.3}$$

$$\{j_n(1)\} = 2, 1, 5, 7, 17, 31, 65, 127, \ldots. \tag{1.4}$$

Binet Forms

From the characteristic equation $\lambda^2 - \lambda - 2x = 0$ for both (1.1) and (1.2), we deduce the roots

$$\alpha = \frac{1+\Delta}{2}, \quad \beta = \frac{1-\Delta}{2}, \tag{1.5}$$

so that

$$\alpha + \beta = 1, \ \alpha\beta = 2x, \ \alpha - \beta = \sqrt{1+8x} = \Delta. \tag{1.6}$$

Binet forms are then

$$J_n(x) = (\alpha^n - \beta^n)/\Delta, \tag{1.7}$$

$$j_n(x) = \alpha^n + \beta^n. \tag{1.8}$$

Generating Functions

$$\sum_{n=0}^{\infty} J_{n+1}(x)y^n = (1 - y - 2xy^2)^{-1}, \tag{1.9}$$

$$\sum_{n=0}^{\infty} j_{n+1}(x)y^n = (1+4xy)(1 - y - 2xy^2)^{-1}. \tag{1.10}$$

An immediate consequence of (1.9) and (1.10) is

$$j_n(x) = J_n(x) + 4xJ_{n-1}(x), \tag{1.11}$$

which is also quickly obtainable from (1.7) and (1.8).

Jacobsthal convolution polynomials $J_n^{(k)}(x)$ are defined [see (4.9) and (4.9a)] from (1.9) by

$$\sum_{n=0}^{\infty} J_{n+1}^{(k)}(x)y^n = (1 - y - 2xy^2)^{-(k+1)}. \tag{1.12}$$

The corresponding Jacobsthal-Lucas convolution polynomials $j_{n+1}^{(k)}(x)y^n$ are defined in (5.7) and (5.7a) by means of (1.10).

2. FIRST JACOBSTHAL CONVOLUTION POLYNOMIALS $J_n^{(1)}(x)$

Generating Function Definition

$$\sum_{n=0}^{\infty} J_{n+1}^{(1)}(x)y^n = (1 - y - 2xy^2)^{-2} \tag{2.1}$$

$$= \left(\sum_{r=0}^{\infty} J_{r+1}(x)y^r\right)^2 \quad \text{by (1.9).} \tag{2.1a}$$

Examples

$$\begin{aligned} &J_1^{(1)}(x) = 1,\ J_2^{(1)}(x) = 2,\ J_3^{(1)}(x) = 3 + 4x,\ J_4^{(1)}(x) = 4 + 12x,\ J_5^{(1)}(x) = 5 + 24x + 12x^2, \\ &J_6^{(1)}(x) = 6 + 40x + 48x^2,\ J_7^{(1)}(x) = 7 + 60x + 120x^2 + 32x^3, \ldots. \end{aligned} \tag{2.2}$$

Special Case (First Jacobsthal Convolution Numbers: $x = 1$)

$$\{J_n^{(1)}(1)\} = 1, 2, 7, 16, 41, 94, 219, \ldots. \tag{2.3}$$

Observe that this sequence of integers appears in the second column of the matrix in [3, p. 401].

Recurrence Relations

Immediately, from (1.9) and (2.1), we deduce the recurrence

$$J_{n+1}^{(1)}(x) - J_n^{(1)}(x) - 2xJ_{n-1}^{(1)}(x) = J_{n+1}(x). \tag{2.4}$$

By means of (2.4), the list of first convolution polynomials may be extended indefinitely.

Partial differentiation with respect to y of both sides of (1.9) along with the equating of the coefficients of y^{n-1} then yields, with (2.1),

$$nJ_{n+1}(x) = J_n^{(1)}(x) + 4xJ_{n-1}^{(1)}(x). \tag{2.5}$$

Combine (2.4) with (2.5) to obtain the recurrence

$$nJ_{n+1}^{(1)}(x) = (n+1)J_n^{(1)}(x) + 2x(n+2)J_{n-1}^{(1)}(x). \tag{2.6}$$

Eliminate $J_{n-1}^{(1)}(x)$ from (2.4) and (2.5). Then

$$(n+2)J_{n+1}(x) = 2J_{n+1}^{(1)}(x) - J_n^{(1)}(x). \tag{2.7}$$

Add (2.5) to (2.7), whence

$$(n+1)J_{n+1}(x) = J_{n+1}^{(1)}(x) + 2xJ_{n-1}^{(1)}(x). \tag{2.8}$$

Or, apply (2.9) below twice with reliance on (3.13), (3.12), and (1.2) in [6] and appeal to the (new) result, $j_{n+1}(x) + 4xj_n(x) = \Delta^2 J_{n+1}(x)$ obtained from Binet forms (1.7) and (1.8) above.

Other Main Properties

Next, we are able to derive the revealing connective relation

$$J_n^{(1)}(x) = \frac{nj_{n+1}(x) + 4xJ_n(x)}{\Delta^2}, \tag{2.9}$$

where Δ is given in (1.6). As a prelude to (2.9), we require the recursion

$$nj_{n+1}(x) = (1+4x)J_n^{(1)}(x) + 4xJ_{n-1}^{(1)}(x) + 8x^2 J_{n-2}^{(1)}(x). \tag{2.10}$$

Establishing (2.10) merely asks us to differentiate (1.10) partially with respect to y, and then perform appropriate algebraic interpretations involving (2.1). Corresponding coefficients of y^{n-1} are then equated.

Proofs of (2.9):

(a) **Induction.** The formula is verifiably valid for $n = 1, 2, 3, 4, 5$. Employing the induction method in conjunction with (2.4) leads us to the desired end.

(b) Alternatively (cf. [8, p. 61, (4.7)]), algebraic manipulation in (2.1) gives

$$\begin{aligned}\sum_{n=1}^{\infty} J_n^{(1)}(x)y^{n-1} &= \frac{(1+4x+4xy+8x^2y^2) + 4x(1-y-2xy^2)}{(1+8x)(1-y-2xy^2)^2} \\ &= \frac{1}{1+8x}\sum_{n=1}^{\infty}(nj_{n+1}(x) + 4xJ_n(x))y^{n-1} \quad \text{by (1.9), (1.10), (2.10).}\end{aligned}$$

Compare coefficients of y^{n-1} and (2.9) ensues.

Observe that a Binet form may be deduced for $J_n^{(1)}(x)$ from (2.9) by means of (1.7) and (1.8). Worth noting in passing is that by combining (1.1) and [6, (3.12)] we may express the numerator of the right-hand side of (2.9) neatly as $(n+1)j_{n+1}(x)-J_{n+1}(x)$.

Explicit Combinatorial Form

Theorem 1:

$$J_n^{(1)}(x)=\sum_{r=0}^{\left[\frac{n-1}{2}\right]}\binom{n-r}{1}\binom{n-r-1}{r}(2x)^r \quad \text{(closed form)}. \tag{2.11}$$

Proof (by induction): Using (2.2), we readily verify that the theorem is true for all $n=1$, 2, 3. Assume it is true for all $n\le N$, that is,

$$\text{Assumption: } J_N^{(1)}(x)=\sum_{r=0}^{\left[\frac{N-1}{2}\right]}\binom{N-r}{1}\binom{N-r-1}{r}(2x)^r. \tag{A}$$

Then the right-hand side of (2.6) becomes

$$N(J_N^{(1)}(x)+2xJ_{N-1}^{(1)}(x))+(J_N^{(1)}(x)+4xJ_{N-1}^{(1)}(x))$$

$$=N\sum_{r=0}^{\left[\frac{N}{2}\right]}(N-r)\binom{N-r}{r}(2x)^r+N\sum_{r=0}^{\left[\frac{N}{2}\right]}\binom{N-r}{r}(2x)^r \quad \text{from (A), on simplifying}$$

$$=N\sum_{r=0}^{\left[\frac{N}{2}\right]}(N-r+1)\binom{N-r}{r}(2x)^r \tag{B}$$

$$=NJ_{N+1}^{(1)}(x), \tag{C}$$

which must be the left-hand side of (2.6).

Consequently, (B) and (C) with (A) show that (2.11) is true for $n=N+1$ and thus for all n. Hence, Theorem 1 is completely demonstrated.

Remarks: Recourse is required in the proof to the use of

(i) N even, N odd considered separately (for convenience),

(ii) Pascal's Formula, and

(iii) the combinatorial result (readily computable)

$$(N-r)\binom{N-r-1}{r}+2(N-r)\binom{N-r-1}{r-1}=N\binom{N-r}{r}. \tag{2.11a}$$

Summation

From (2.4) and [6, (3.7)],

$$\sum_{r=1}^{n}J_r^{(1)}(x)=\frac{2xJ_{n+2}^{(1)}(x)-J_{n+4}(x)+1}{4x^2}. \tag{2.12}$$

Expanding the right-hand side of (2.1a), both sides having lower bound $n=1$, and equating coefficients, we arrive at

$$J_n^{(1)}(x)=\begin{cases}2\sum_{r=1}^{[\frac{n}{2}]} J_r(x)J_{n-r+1}(x) & n \text{ even},\\ 2\sum_{r=1}^{[\frac{n-1}{2}]} J_r(x)J_{n-r+1}(x)+J^2_{\frac{n+1}{2}}(x) & n \text{ odd}.\end{cases} \tag{2.13}$$

Differentiation and Convolutions

Let the prime ($'$) represent partial differentiation with respect to x. Differentiate both sides of (1.9) with respect to x. Compare this with (2.1). Then, on equating coefficients of y^{n-1}, we deduce the notably succinct connection

$$2J_{n-1}^{(1)}(x)=J'_{n+1}(x). \tag{2.14}$$

But $j'_n(x)=2nJ_{n-1}(x)$ by [6, (3.21)]. Hence, the second derivative is

$$j''_n(x)=4nJ_{n-3}^{(1)}(x). \tag{2.15}$$

3. FIRST JACOBSTHAL-LUCAS CONVOLUTION POLYNOMIALS $j_n^{(1)}(x)$

Generating Function Definition

$$\sum_{n=0}^{\infty} j_{n+1}^{(1)}(x)y^n = (1+4xy)^2(1-y-2xy^2)^{-2} \tag{3.1}$$

$$= \left(\sum_{r=0}^{\infty} j_{r+1}(x)y^r\right)^2 \quad \text{by (1.10)}. \tag{3.1a}$$

Examples:

$$\begin{aligned}&j_1^{(1)}(x)=1,\ j_2^{(1)}(x)=2+8x,\ j_3^{(1)}(x)=3+20x+16x^2,\ j_4^{(1)}(x)=4+36x+64x^2,\\ &j_5^{(1)}(x)=5+56x+156x^2+64x^3,\ j_6^{(1)}(x)=6+80x+304x^2+228x^3,\dots\end{aligned} \tag{3.2}$$

Special Case (First Jacobsthal-Lucas Convolution Numbers: $x=1$)

$$\{j_1^{(1)}(1)\}=1, 10, 39, 104, 281, 678, 1627,\dots. \tag{3.3}$$

Recurrence Relations

Immediately, from (2.1) and (3.1), we have

$$j_n^{(1)}(x)=J_n^{(1)}(x)+8xJ_{n-1}^{(1)}(x)+16x^2J_{n-2}^{(1)}(x), \tag{3.4}$$

by means of which a list of convolution polynomials may be presented, in conjunction with (2.2), which may be checked against those already given in (3.2).

Combining (3.4) and (2.10), we deduce that

$$2nj_{n+1}(x)=j_n^{(1)}(x)+(1+8x)J_n^{(1)}(x) \quad (1+8x=\Delta^2). \tag{3.5}$$

Equations (2.9) and (3.5) generate the pleasing connection

$$j_n^{(1)}(x)=nj_{n+1}(x)-4xJ_n(x), \tag{3.6}$$

which, with (1.11), may be cast in the form

$$(n-1)j_{n+1}(x)=j_n^{(1)}(x)-J_{n+1}(x). \tag{3.7}$$

Alternatively, (3.6) may be demonstrated in the following way.

$$\begin{aligned}\sum_{n=1}^{\infty} j_n^{(1)}(x)y^{n-1} &= (1+4xy)\cdot\frac{1+4xy}{(1-y-2xy^2)^2} \quad \text{by (3.1)}\\ &= (1+4xy)\sum_{n=1}^{\infty} nJ_{n+1}(x)y^{n-1} \quad \text{differentiating (1.9) w.r.t. } y\\ &= \sum_{n=1}^{\infty}(nJ_{n+1}(x)y^{n-1}+4x(n-1)J_n(x))y^{n-1},\end{aligned}$$

whence (3.6) emerges by (1.11).

Other Main Properties

Comparing the generating functions in (1.10) and (2.1), we calculate upon simplification that

$$j_n(x) = J_n^{(1)}(x)+(4x-1)J_{n-1}^{(1)}(x)-64xJ_{n-2}^{(1)}(x)-8x^2J_{n-3}(x). \tag{3.8}$$

Taken together, (2.9) and (3.6) produce

$$J_n^{(1)}(x)j_n^{(1)}(x) = \frac{n^2 j_{n+1}^2(x)-16x^2J_n^2(x)}{\Delta^2} \quad (\Delta^2 = 1+8x). \tag{3.9}$$

Equation (3.6), in conjunction with (1.7) and (1.8), allows us to display $j_n^{(1)}(x)$ in a Binet form.

Furthermore, (2.9) and (3.6) yield

$$\Delta^2 J_n^{(1)}(x)+j_n^{(1)}(x) = 2nj_{n+1}(x) \tag{3.10}$$

and

$$\Delta^2 J_n^{(1)}(x)-j_n^{(1)}(x) = 8xJ_n(x). \tag{3.11}$$

Lastly, we append a result which is left as an exercise for the curiosity of the reader:

$$(\Delta^2-1)j_n(x) = \Delta^2\{J_{n+1}^{(1)}(x)+2xJ_{n-1}^{(1)}(x)\}-\{j_{n+1}^{(1)}(x)+2xj_{n-1}^{(1)}(x)\}, \tag{3.12}$$

where $\Delta^2-1=8x$ by (1.6).

4. GENERAL JACOBSTHAL CONVOLUTION POLYNOMIALS $J_n^{(k)}(x)$ $(k>1)$

A. CASE $k=2$ (Second Jacobsthal Convolution Polynomials)

Generating Function Definition

$$\sum_{n=0}^{\infty} J_{n+1}^{(2)}(x)y^n = (1-y-2xy^2)^{-3} \tag{4.1}$$

$$= \left(\sum_{r=0}^{\infty} J_{r+1}(x)y^r\right)^3. \tag{4.1a}$$

Examples

$$\begin{aligned}&J_1^{(2)}(x)=1,\ J_2^{(2)}(x)=3,\ J_3^{(2)}(x)=6+6x,\ J_4^{(2)}(x)=10+24x,\ J_5^{(2)}(x)=15+60x+24x^2,\\ &J_6^{(2)}(x)=21+120x+120x^2,\ J_7^{(2)}(x)=28+210x+360x^2+80x^3,\ldots.\end{aligned} \tag{4.2}$$

Special Case (Second Jacobsthal Convolution Numbers: $x=1$)

$$\{J_n^{(2)}(1)\} = 1, 3, 12, 34, 99, 261, 678, \ldots. \tag{4.3}$$

Observe that this sequence of numbers occurs in the third column of the matrix array in [3, p. 401].

Recurrence Relations

Immediately, from (2.1) and (4.1) there comes

$$J_{n+1}^{(2)}(x) - J_n^{(2)}(x) - 2xJ_{n-1}^{(2)}(x) = J_{n+1}^{(1)}(x) \tag{4.4}$$

whereas (1.9) and (4.1) lead to

$$J_{n+1}^{(2)}(x) - 2J_n^{(2)}(x) + (1-4x)J_{n-1}^{(2)}(x) + 4xJ_{n-2}^{(2)}(x) + 4x^2J_{n-3}^{(2)}(x) = J_{n+1}(x). \tag{4.5}$$

Differentiate both sides of (2.1) partially with respect to y, then equate coefficients of y^{n-1} to obtain, by (4.1),

$$nJ_{n+1}^{(1)}(x) = 2(J_n^{(2)}(x) + 4xJ_{n-1}^{(2)}(x)). \tag{4.6}$$

Eliminate $J_{n+1}^{(1)}(x)$ from (4.4) and (4.6). Hence,

$$nJ_{n+1}^{(2)}(x) = (n+2)J_n^{(2)}(x) + 2x(n+4)J_{n-1}^{(2)}(x). \tag{4.7}$$

Next, eliminate $J_{n-1}^{(2)}(x)$ from (4.4) and (4.6). Accordingly,

$$(n+4)J_{n+1}^{(1)}(x) = 2(2J_{n+1}^{(2)}(x) - J_n^{(2)}(x)). \tag{4.8}$$

Not all results in Section 3 above ($k=1$) extend readily to direct unit superscript increase on both sides of the equation [cf. (2.7), (4.8)].

B. CASE k General (k^{th} Jacobsthal Convolution Polynomials)

Generating Function Definition

$$\sum_{n=0}^{\infty} J_{n+1}^{(k)}(x)y^n = (1-y-2xy^2)^{-(k+1)} \tag{4.9}$$

$$= \left(\sum_{r=0}^{\infty} J_{r+1}(x)y^2\right)^{k+1} \quad \text{by (1.9).} \tag{4.9a}$$

Examples

$$\left.\begin{aligned} &J_1^{(k)}(x) = 1,\ J_2^{(k)}(x) = \binom{k+1}{1},\ J_3^{(k)}(x) = \binom{k+2}{2} + \binom{k+1}{1}2x, \\ &J_4^{(k)}(x) = \binom{k+3}{3} + \binom{k+2}{2}4x,\ J_5^{(k)}(x) = \binom{k+4}{4} + \binom{k+3}{3}\cdot 3\cdot 2x + \binom{k+2}{2}(2x)^2, \ldots \end{aligned}\right. \tag{4.10}$$

Special Case (k^{th} Jacobsthal Convolution Numbers: $x=1$)

$$\{J_n^{(k)}(1)\} = 1,\ k+1,\ (k+1)\left(\frac{k+6}{2}\right),\ (k+1)(k+2)\left(\frac{k+15}{6}\right), \ldots. \tag{4.11}$$

Explicit Combinatorial Form

Theorem 2:

$$J_n^{(k)}(x) = \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \binom{k+n-r-1}{k}\binom{n-r-1}{r}(2x)^r. \tag{4.12}$$

Proof: Constructing the proof parallels the procedures employed in Theorem 1, where $k = 1$. That is, apply (4.15), which will be proven independently below, and induction in tandem.

Remarks: Corresponding to the combinatorial identity (2.11a) for Theorem 1, we require in the proof of Theorem 2,

$$\begin{aligned} &k\left\{\binom{N+k-1-r}{k}\binom{N-r-1}{r}+2\binom{N+k-1-r}{k}\binom{N-r-1}{r-1}\right\} \\ &= N\binom{N+k-1-r}{k-1}\binom{N-r}{r}, \end{aligned} \tag{4.12a}$$

i.e., k is absorbed into the product and N emerges as a factor.

Finally, we have the sum

$$\begin{aligned} &N\left[\binom{N+k-1-r}{k}\binom{N-r}{r}+\binom{N+k-1-r}{k-1}\binom{N-r}{r}\right] \\ &= N\binom{N+k-r}{k}\binom{N-r}{r}. \end{aligned} \tag{4.12b}$$

Pascal's formula is needed in (4.12a) and (4.12b). The simplified form in (4.12b) relates to the expression for $J_{N+1}^{(k)}(x)$ in (4.12).

Knowledge of (4.12) now permits us to compute $J_n^{(k)}(x)$ for any k and n. In particular, $J_5^{(3)}(x) = 35 + 120x + 40x^2$. Refer also to (4.10).

Recurrence Relations

Appealing to (4.9) and (4.9) with $k-1$, we have the immediate consequence

$$J_{n+1}^{(k)}(x) - J_n^{(k)}(x) - 2xJ_{n-1}^{(k)}(x) = J_{n+1}^{(k-1)}(x). \tag{4.13}$$

Partially differentiate both sides of (4.9) with respect to y. Considering coefficients of y^{n-1} we then have, on replacing k by $k-1$,

$$nJ_{n+1}^{(k-1)}(x) = k(J_n^{(k)}(x) + 4xJ_{n-1}^{(k)}(x)). \tag{4.14}$$

Combine (4.13) and (4.14) to obtain the recurrence

$$nJ_{n+1}^{(k)}(x) = (n+k)J_n^{(k)}(x) + 2x(n+2k)J_{n-1}^{(k)}(x). \tag{4.15}$$

Furthermore, from (4.13) and (4.14), we arrive at

$$(n+2k)J_{n+1}^{(k-1)}(x) = k(2J_{n+1}^{(k)}(x) - J_n^{(k)}(x)). \tag{4.16}$$

Results when $k = 2$ may now be checked against those specialized in (4.1)-(4.8).

Convolution Array for $J_n^{(k)}$

In Table 1 below, we exhibit the simplest numbers occurring in the Jacobsthal array for the convolution numbers $J_n^{(k)}$.

Convolution numbers for $k = 1, 2$ and for small values of n are already publicized in (2.3), (4.3) and (3.3), (5.3). Applying the extremely useful formulas obtained (from the Cauchy convolutions of a sequence with itself) by induction in [1, pp. 193-94], where the initial conditions (1.1), (1.2) are known, we may develop the array for $J_n^{(k)}$ to our heart's desire. Or use Theorem 2 when

$x=1$. Systematic reduction to $n=1$ (boundary case) using (4.13) is a rewarding, if tedious, exercise. Reduction by (4.13) gives, for example, $J_4^{(2)}(x)=10+24x$ in conformity with (4.2).

TABLE 1. Convolution Array for $J_n^{(k)}$ $(n=1,2,\dots,5)$

n/k	0	1	2	3	$\cdots$	k
1	1	1	1	1	$\cdots$	1
2	1	2	3	4	$\cdots$	$\binom{k+1}{1}$
3	3	7	12	18	$\cdots$	$\binom{k+2}{2}+2\binom{k+1}{1}$
4	5	16	34	60	$\cdots$	$\binom{k+3}{3}+4\binom{k+2}{2}$
5	11	41	99	195	$\cdots$	$\binom{k+4}{4}+6\binom{k+3}{3}+4\binom{k+2}{2}$

It should be noted that the formulas given in [1, pp. 193-94] relate to rows in the convolution array, whereas it is the columns that are generated in our approach, namely, one column for each convolution value of k.

Be aware that the notation in [1, pp. 193-94] is different, namely, we have the correspondence (subscripts in R_{nk} referring to rows and columns, respectively)

$$R_{nk} \Leftrightarrow J_n^{(k-1)}. \tag{4.17}$$

Formula (4.10) and [1, (1.6)] then both yield, for example, $R_{43}=J_4^{(2)}=34$ (Table 1).

Reverting briefly to [3, p. 401] we see that the abbreviated array for $J_n^{(k)}$ is exposed in matrix form in which the first, second, third, ... columns of the matrix B_2P are precisely our $J_n^{(0)}$, $J_n^{(1)}$, $J_n^{(2)}$, ..., respectively. *En passant*, we remark that the columns of the matrix A_2P are exactly the Pell convolution numbers $P_n^{(0)}$, $P_n^{(1)}$, $P_n^{(2)}$, ... examined in [8].

5. GENERAL JACOBSTHAL-LUCAS CONVOLUTION POLYNOMIALS $j_n^{(k)}(x)$ $(k>1)$

A. CASE $k=2$ (Second Jacobsthal-Lucas Convolution Polynomials)

Generating Function Definition

$$\sum_{n=0}^{\infty} j_{n+1}^{(2)}(x)y^n = (1+4xy)^3[1-y-2xy^2]^{-3} \tag{5.1}$$

$$= \left(\sum_{r=0}^{\infty} j_{r+1}(x)y^r\right)^3. \tag{5.1a}$$

Examples

$$\begin{aligned} &j_1^{(2)}(x)=1,\ j_2^{(2)}(x)=3+12x,\ j_3^{(2)}(x)=6+42x+48x^2,\\ &j_4^{(2)}(x)=10+96x+216x^2+64x^3,\ j_5^{(2)}(x)=15+180x+600x^2+480x^3,\dots. \end{aligned} \tag{5.2}$$

Special Case (Second Jacobsthal-Lucas Convolution Numbers: $x=1$)

$$\{j_n^{(2)}(1)\}=1, 15, 96, 386, 1275, \dots.$$

Recurrence Relations

Taken together, (1.10), (3.1), and (5.1) yield

$$\sum_{n=0}^{\infty} j_{n+1}^{(2)}(x)y^n = \left(\sum_{n=0}^{\infty} j_{n+1}(x)y^n\right)\left(\sum_{n=0}^{\infty} j_{n+1}^{(1)}(x)y^n\right). \tag{5.4}$$

Comparing coefficients of y^n, we deduce that

$$j_{n+1}^{(2)}(x) = \sum_{r=1}^{n+1} j_r(x) j_{n-r+2}^{(1)}(x). \tag{5.5}$$

Furthermore, from (4.1) and (5.1), we easily derive

$$j_n^{(2)}(x) = J_n^{(2)}(x) + 12xJ_{n-1}^{(2)}(x) + 48x^2 J_{n-2}^{(2)}(x) + 64x^3 J_{n-3}^{(2)}(x). \tag{5.6}$$

B. CASE k General (k^{th} Jacobsthal-Lucas Convolution Polynomials)

Generating Function Definition

$$\sum_{n=0}^{\infty} j_{n+1}^{(k)}(x)y^n = (1+4xy)^{k+1}[1-y-2xy^2]^{-(k+1)} \tag{5.7}$$

$$= \left(\sum_{r=0}^{\infty} j_{r+1}(x)y^r\right)^{k+1}. \tag{5.7a}$$

Examples

$$j_1^{(k)}(x) = 1,\ j_2^{(k)}(x) = \binom{k+1}{1}(1+4x),$$
$$j_3^{(k)}(x) = \binom{k+1}{2}16x^2 + 2x\binom{k+1}{1}\left\{2\binom{k+1}{1}+1\right\} + \binom{k+2}{2}, \ldots. \tag{5.8}$$

Special Case (k^{th} Jacobsthal-Lucas Convolution Numbers: $x = 1$)

$$\{j_n^{(k)}(1)\} = 1,\ 5\binom{k+1}{1},\ 16\binom{k+1}{2} + 2\binom{k+1}{1}\left\{2\binom{k+1}{1}+1\right\} + \binom{k+2}{2}, \ldots. \tag{5.9}$$

Theorem 3:

$$j_n^{(k)}(x) = \sum_{r=0}^{k+1}\binom{k+1}{r}(4x)^r J_{n-r}^{(k)}(x), \tag{5.10}$$

where $J_{n-r}^{(k)}(x)$ are given in (4.12).

Proof: Expand $(1+4xy)^{k+1}$ in conjunction with (4.9) and (5.7) to produce

$$j_n^{(k)}(x) = J_n^{(k)}(x) + \binom{k+1}{1}4xJ_{n-1}^{(k)}(x) + \binom{k+1}{2}(4x)^2 J_{n-2}^{(k)}(x) + \cdots$$
$$+ \binom{k+1}{r}(4x)^r J_{n-r}^{(k)}(x) + \cdots + (4x)^{k+1} J_{n-k-1}^{(k)}(x).$$

The theorem is thus demonstrated.

Armed with this knowledge (5.10), we may then appeal to (4.12) for the determination of the convolution polynomials $j_n^{(k)}(x)$ for any k and n. For example, application of (5.10) leads us to $j_5^{(2)}(x) = 15 + 180x + 600x^2 + 480x^3$, which confirms (5.2).

Convolution Array for $j_n^{(k)}$

A truncated array for $j_n^{(k)}$ is set out in Table 2.

TABLE 2. Convolution Array for $j_n^{(k)}$ $(n = 1, 2, \ldots, 5)$

$n\,/\,k$	0	1	2	3	$\cdots$	k
1	1	1	1	1	$\cdots$	1
2	5	10	15	20	$\cdots$	$5\binom{k+1}{1}$
3	7	39	96	178	$\cdots$	$16\binom{k+1}{2}+\binom{k+1}{1}\left\{4\binom{k+1}{1}+2\right\}+\binom{k+2}{2}$
4	17	104	386	488	$\cdots$	$\cdots$
5	31	281	1275	4163	$\cdots$	$\cdots$

As in (4.16), we have the correspondence of notation

$$R_{nk} \Leftrightarrow j_n^{(k-1)}, \tag{5.11}$$

where subscripts in R_{nk} refer to rows and columns, respectively, whence, for instance, $R_{32} = j_3^{(1)} = 39$ (Table 2).

Evidently, there is a law of diminishing returns evolving as we proceed to study the case for k general, and more so as we progress from $J_n^{(k)}(x)$ to $j_n^{(k)}(x)$. Perhaps we should follow a precept of Descartes and leave further discoveries for the pleasure of the assiduous investigator.

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