FIBONACCI-LUCAS QUASI-CYCLIC MATRICES

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1. INTRODUCTION

Matrices such as

$$R = R(D; x_1, x_2, \ldots, x_n) = \begin{pmatrix} x_1 & Dx_n & \cdots & Dx_2 \\ x_2 & x_1 & \cdots & Dx_3 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{pmatrix} \tag{1}$$

are called quasi-cyclic matrices. These matrices were introduced and studied in [2] and [5]. We can obtain these matrices by multiplying every element of the upper triangular part (not including the diagonal) of the cyclic matrices (see [4])

$$C = \begin{pmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \cdots & x_3 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{pmatrix}$$

by D.

In this paper we will prove that, for $n \geq 2$,

$$\det(R(L_n; F_{2n-1}, F_{2n-2}, \ldots, F_n)) = 1,$$

where L_n and F_n denote, as usual, the n^{th} Lucas and Fibonacci numbers, respectively, and $\det(R)$ denotes the determinant of R. In addition, if we let

$$R_{n,k} = R(L_n; F_{2n-1+k}, F_{2n-2+k}, \ldots, F_{n+k})$$

for integral k, then

$$\det(R_{n,k}) = (-1)^{n-1} L_n F_k^n + F_{k-1}^n.$$

The motivation for studying these determinants comes from Pell's equation. It is well known that the solution of Pell's equation $x^2 - dy^2 = \pm 1$ is closely related to the unit of the quadratic field $Q(\sqrt{d})$. We may extend the conclusion to fields of higher degree. If we rewrite $x^2 - dy^2 = \pm 1$ as

$$\det\begin{pmatrix} x & dy \\ y & x \end{pmatrix} = \pm 1,$$

we can easily do this. The equation

$$\det\begin{pmatrix} x_1 & Dx_n & \cdots & Dx_2 \\ x_2 & x_1 & \cdots & Dx_3 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & \cdots & x_1 \end{pmatrix} = \pm 1 \tag{2}$$

is called Pell's equation of degree n. Using our results, we can obtain solutions to an infinite family of Pell equations of higher degree based on Fibonacci and Lucas numbers.

To prove our results, we will need two propositions. These two propositions came from [2] and [5].

Proposition 1:

$$\det(R)=\prod_{k=0}^{n-1}\left(\sum_{i=1}^{n}x_i d^{i-1}\varepsilon^{k(i-1)}\right), \tag{3}$$

where $d=\sqrt[n]{D}$ and $\varepsilon=e^{2\pi i/n}$. Also, each factor $\sum_{i=1}^{n}x_i d^{i-1}\varepsilon^{k(i-1)}$ of the right-hand side of (3) is an eigenvalue of the matrix R.

Proposition 2: Let n and D be fixed. Then the sum, difference, and product of two quasi-cyclic matrices is also quasi-cyclic. The inverse of a quasi-cyclic matrix is quasi-cyclic.

2. THE MAIN RESULTS AND THEIR PROOFS

We are now ready to state and prove the first theorem.

Theorem 1: Let $n\geq 2$. Then

$$\det(R(L_n;F_{2n-1},F_{2n-2},\dots,F_n))=1, \tag{4}$$

where L_n and F_n denote, as usual, the n^{th} Lucas and Fibonacci numbers, respectively.

Proof: For $n=2$, we have that

$$\det(R(L_2;F_3,F_2))=\det\begin{pmatrix}2&3\\1&2\end{pmatrix}=1,$$

so the result of the theorem holds. If $n>2$, let

$$T=\begin{pmatrix}1&-1&-1&0&\cdots&0&0\\0&1&-1&-1&\cdots&0&0\\0&0&1&-1&\cdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&\cdots&1&-1\\0&0&0&0&\cdots&0&1\end{pmatrix}. \tag{5}$$

By multiplication of matrices and properties of Fibonacci and Lucas numbers, we have

$$RT=\begin{pmatrix}F_{2n-1}&F_{2n-2}&(-1)^n&&\\\vdots&\vdots&&\ddots&\\\vdots&\vdots&&&(-1)^n\\F_{n+1}&F_n&0&\cdots&0\\F_n&F_{n-1}&0&\cdots&0\end{pmatrix}. \tag{6}$$

Taking the determinant of both sides of (6) and noting that $\det(T)=1$, we have

$$\begin{aligned}\det(R)&=\det(R)\det(T)=\det(RT)\\&=(-1)^{2n-4}\det\begin{pmatrix}F_{n+1}&F_n&0&\cdots&0\\F_n&F_{n-1}&0&\cdots&0\\F_{2n-1}&F_{2n-2}&&&\\\vdots&\vdots&&(-1)^nI_{n-2}&\\F_{n+2}&F_{n+1}&&&\end{pmatrix}\end{aligned}$$

$$= \det\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \det((-1)^n I_{n-2})$$
$$= (F_{n+1}F_{n-1} - F_n^2)(-1)^{n(n-2)} = (-1)^n(-1)^n = 1,$$

where I_n denotes the identity matrix of order n. Thus, Theorem 1 is true.

Corollary 1: If $D = L_n$, then $(F_{2n-1}, F_{2n-2}, \dots, F_n)$ is a solution of Pell's equation (3).

Corollary 2: Let $d = \sqrt[n]{L_n}$, $\varepsilon = e^{2\pi i/n}$. Then

$$\prod_{k=0}^{n-1}\left(\sum_{i=1}^{n} F_{2n-i} d^{i-1} \varepsilon^{k(i-1)}\right) = 1.$$

Proof: This is obvious by Theorem 1 and Proposition 1.

We now make the following conclusion.

Theorem 2: The matrix $R = R(L_n; F_{2n-1}, \dots, F_n)$ is invertible. In addition,

$$R^{-1} = (-1)^{n-1}(I + E - E^2), \tag{7}$$

where $I = I_n$ and

$$E = E_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & L_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Proof: Since $\det(R) = 1 \neq 0$, the inverse R^{-1} exists. Obviously,

$$R = R(L_n; F_{2n-1}, \dots, F_n) = F_{2n-1}I + F_{2n-2}E + F_{2n-3}E^2 + \cdots + F_nE^{n-1}.$$

Hence,

$$\begin{aligned}
&R(-1)^{n-1}(I + E - E^2) \\
&= (-1)^{n-1}(F_{2n-1}I + F_{2n-2}E + F_{2n-3}E^2 + \cdots + F_nE^{n-1} + F_{2n-1}E + F_{2n-2}E^2 + \cdots \\
&\qquad + F_{n+1}E^{n-1} + F_nE^n - F_{2n-1}E^2 - \cdots - F_{n+2}E^{n-1} - F_{n+1}E^n - F_nE^{n+1}) \\
&= (-1)^{n-1}(F_{2n-1}I + F_{2n-2}E + F_{2n-1}E + F_nE^n - F_{n+1}E^n - F_nE^{n+1}) \\
&= (-1)^{n-1}(F_{2n-1}I + F_{2n}E + F_nL_nI - F_{n+1}L_nI - F_nL_nE) \\
&= (-1)^{n-1}(F_{2n-1}I + F_{2n}I + F_{2n-1}E - F_{2n+1}I - (-1)^nI) \\
&= (-1)^{n-1}(-1)^{n+1}I = I.
\end{aligned}$$

In the above, the three following facts have been used:

1. $F_{2n-3} + F_{2n-2} - F_{2n-1} = 0, \dots, F_n + F_{n+1} - F_{n+2} = 0$. This is obvious from the definition of Fibonacci numbers.
2. $E^n = L_nI$, $E^{n+1} = L_nE$. This can be verified easily by multiplication of matrices.
3. $L_nF_n = F_{2n}$ and $L_nF_{n+1} = F_{2n+1} + (-1)^n$. These are well-known properties of Fibonacci and Lucas numbers.

Corollary 3: Let $n \geq 3$ be an odd number and $D = L_n$. Then

$$(x_1, x_2, x_3, x_4, \dots, x_n) = (1, 1, -1, 0, \dots, 0)$$

is a solution of Pell's equation (3) of degree n.

Let $n \geq 4$ be an even number and $D = L_n$. Then

$$(x_1, x_2, x_3, x_4, \dots, x_n) = (-1, -1, 1, 0, \dots, 0)$$

is a solution of Pell's equation (3) of degree n.

Proof: Based on Theorem 2, when n is odd, we have

$$\det(R(L_n; 1, 1, -1, 0, \dots, 0)) \det(R(L_n; F_{2n-1}, \dots, F_n)) = \det(I) = 1$$

and

$$\det(R(L_n; F_{2n-1}, \dots, F_n)) = 1$$

from Theorem 1, so

$$\det(R(L_n; 1, 1, -1, 0, \dots, 0)) = 1$$

and, by definition of solution, the conclusion is true. For even n, the proof is similar.

3. MORE RESULTS ABOUT THE DETERMINANTS

Let $R_{n,k} = R(L_n; F_{2n-1+k}, F_{2n-2+k}, \dots, F_{n+k})$, $k = 0, \pm 1, \pm 2, \dots$, be square matrices of degree n. Then Theorem 1 has the form $\det(R_{n,0}) = 1$. For $\det(R_{n,1})$, $\det(R_{n,2}), \dots, \det(R_{n,-1})$, $\det(R_{n,-2})$, ..., we can also obtain corresponding results, but the values of these determinants are not 1, so that the inverses $R_{n,k}^{-1}$ of $R_{n,k}$, $k = \pm 1, \pm 2$, ..., are not matrices with integer elements.

Theorem 3:

$$\begin{aligned}
\det(R_{n,-2}) &= 2^n - L_n, \\
\det(R_{n,-1}) &= (-1)^{n-1}(L_n - 1), \\
\det(R_{n,0}) &= 1, \\
\det(R_{n,1}) &= (-1)^{n-1} L_n, \\
\det(R_{n,2}) &= (-1)^{n-1} L_n + 1.
\end{aligned}$$

The result in the middle of Theorem 3, i.e., $\det(R_{n,0}) = 1$ is just Theorem 1. The other results are closely related to L_n, so we list them here. In fact, they can be deduced from the more extensive following results.

Theorem 4: Let $n \geq 2$ be an integer and let k be an integer. Then

$$\det(R_{n,k}) = (-1)^{n-1} L_n F_k^n + F_{k-1}^n.$$

To prove Theorem 4, set

$$g_{n,k} = \begin{vmatrix}
F_{2n+k-1} & F_{2n+k-2} & (-1)^n F_{k-1} & & \\
F_{2n+k-2} & F_{2n+k-3} & (-1)^n F_k & \ddots & \\
\vdots & \vdots & & \ddots & (-1)^n F_{k-1} \\
\vdots & \vdots & & & (-1)^n F_k \\
F_{n+k} & F_{n+k-1} & 0 & \cdots & 0
\end{vmatrix}, \tag{8}$$

$$h_{n,k}=\begin{vmatrix} F_{2n+k-1} & (-1)^nF_k & (-1)^nF_{k-1} & & \\ F_{2n+k-2} & 0 & (-1)^nF_k & \ddots & \\ \vdots & \vdots & & \ddots & (-1)^nF_{k-1} \\ \vdots & \vdots & & & (-1)^nF_k \\ F_{n+k} & 0 & 0 & \cdots & 0 \end{vmatrix}, \tag{9}$$

where the elements in the middle are zero in every determinant. Now the proof of Theorem 4 consists of the following four points:

1. $\det(R_{n,k})=g_{n,k}+h_{n,k}$;
2. $g_{n,k}=F_{k-1}^n+(-1)^{n-1}F_{n-1}F_k^n+(-1)^nF_k^{n-1}F_nF_{k-1}$;
3. $h_{n,k}=(-1)^{n-1}F_{n+k}F_k^{n-1}$;
4. $g_{n,k}+h_{n,k}=(-1)^{n-1}L_nF_k^n+F_{k-1}^n$.

We can obtain the above four points from five lemmas.

Lemma 1: Suppose $g_{n,k}$ and $h_{n,k}$ are defined as in (8) and (9). Then $\det(R_{n,k})=g_{n,k}+h_{n,k}$.

Proof: Let T be as in (5). Then, by properties of determinants, we have

$$\begin{aligned}\det(R_{n,k}) &= \det(R_{n,k})\det(T)\\ &= \det(R_{n,k}\cdot T)\\ &= \begin{vmatrix} F_{2n+k-1} & F_{2n+k-2}+(-1)^nF_k & (-1)^nF_{k-1} & & \\ F_{2n+k-2} & F_{2n+k-3} & (-1)^nF_k & \ddots & \\ \vdots & \vdots & & \ddots & (-1)^nF_{k-1} \\ \vdots & \vdots & & & (-1)^nF_k \\ F_{n+k} & F_{n+k-1} & 0 & \cdots & 0 \end{vmatrix}\\ &= g_{n,k}+h_{n,k}.\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2 (the recurrence of $g_{n,k}$):

$$g_{n,k}=(-1)^{n+k}F_{n-1}F_k^{n-2}+F_{k-1}g_{n-1,k}. \tag{10}$$

Proof: By subtracting the second column from the first column of $g_{n,k}$, the first column becomes $(F_{2n+k-3},F_{2n+k-4},\dots,F_{n+k-2})^T$ by the properties of Fibonacci numbers, where T in the superscript denotes the transpose of a matrix or vector. By subtracting the first column from the second column, and so on, after $n+k-1$ subtractions between the two columns, the first two columns become

$$\begin{pmatrix} F_{n-1} & F_{n-2} & \cdots & F_0 \\ F_n & F_{n-1} & \cdots & F_1 \end{pmatrix}^T.$$

Next we exchange the first two columns if $n+k$ is even and we keep the matrix if $n+k$ is odd. Hence, the first two columns become

$$\begin{pmatrix} F_n & F_{n-1} & \cdots & F_1 \\ F_{n-1} & F_{n-2} & \cdots & F_0 \end{pmatrix}^T.$$

Thus,

$$g_{n,k} = (-1)^{n+k-1}\begin{vmatrix} F_n & F_{n-1} & (-1)^n F_{k-1} & \ddots & \\ F_{n-1} & F_{n-2} & (-1)^n F_k & \ddots & \\ \vdots & \vdots & & \ddots & (-1)^n F_{k-1} \\ \vdots & \vdots & & & (-1)^n F_k \\ F_1 & F_0 & 0 & \cdots & 0 \end{vmatrix}$$

$$= (-1)^{k-1}\begin{vmatrix} F_n & F_{n-1} & F_{k-1} & \ddots & \\ F_{n-1} & F_{n-2} & F_k & \ddots & \\ \vdots & \vdots & & \ddots & F_{k-1} \\ \vdots & \vdots & & & F_k \\ F_1 & F_0 & 0 & \cdots & 0 \end{vmatrix}.$$

Expanding the last determinant by the first row and noting that $F_0 = 0$, we have

$$g_{n,k} = (-1)^{k-1}\left(-F_{n-1}\begin{vmatrix} F_{n-1} & F_k & F_{k-1} & \ddots & \\ & & F_k & \ddots & \\ \vdots & \vdots & & \ddots & F_{k-1} \\ \vdots & \vdots & & & F_k \\ F_1 & F_0 & 0 & \cdots & 0 \end{vmatrix} + F_{k-1}\begin{vmatrix} F_{n-1} & F_{n-2} & F_{k-1} & \ddots & \\ & & F_k & \ddots & \\ \vdots & \vdots & & \ddots & F_{k-1} \\ \vdots & \vdots & & & F_k \\ F_1 & F_0 & 0 & \cdots & 0 \end{vmatrix} \right)$$

$$= (-1)^{k-1}(-F_{n-1}(-1)^{1+n-1}F_1F_k^{n-2} + F_{k-1}(-1)^{k-1}g_{n-1,k}) = (-1)^{n+k}F_{n-1}F_k^{n-2} + F_k g_{n-1,k}.$$

Thus, Lemma 2 is proved.

Lemma 3:

$$g_{n,k} = F_{k-1}^n + (-1)^{n-1}F_{n-1}F_k^n + (-1)^n F_n F_k^{n-1}F_{k-1}. \tag{11}$$

Proof: By induction on n.

(A) On the one hand, by the definition of $g_{n,k}$, we have

$$g_{2,k} = \begin{vmatrix} F_{3+k} & F_{2+k} \\ F_{2+k} & F_{1+k} \end{vmatrix} = F_{3+k}F_{1+k} - F_{2+k}^2 = (-1)^{k+2-1} = (-1)^{k-1}.$$

On the other hand, the right side of (11) becomes

$$F_{k-1}^2 + (-1)F_{2-1}F_k^2 + (-1)^2F_2F_k^{2-1}F_{k-1} = F_{k-1}^2 - F_k^2 + F_{k-1}F_k$$
$$= F_{k-1}^2 - F_{k-1}F_k - F_k^2 = F_{k-1}F_{k+1} - F_k^2 = (-1)^{k-1}.$$

Hence, Lemma 3 holds when $n = 2$.

(B) Assume (11) holds for $n-1$, i.e.,

$$g_{n-1,k} = F_{k-1}^{n-1} + (-1)^{n-2}F_{n-2}F_k^{n-1} + (-1)^{n-1}F_{n-1}F_k^{n-1}F_{k-1}. \tag{12}$$

We will prove that (11) holds for n. By (12) and recurrence (10), we have

$$g_{n-1,k} = (-1)^{n+k}F_{n-1}F_k^{n-2} + F_{k-1}(F_{k-1}^{n-1} + (-1)^{n-2}F_{n-2}F_k^{n-1} + (-1)^{n-1}F_{n-1}F_k^{n-2}F_{k-1})$$
$$= F_{k-1}^n + (-1)^{n-1}(-1)^{k-1}F_{n-1}F_k^{n-2} + (-1)^{n-2}F_{n-2}F_k^{n-1}F_{k-1} + (-1)^{n-1}F_{n-1}F_k^{n-2}F_{k-1}^2$$
$$= F_{k-1}^n + (-1)^{n-1}F_k^{n-2}((-1)^{k-1}F_{n-1} - (F_n - F_{n-1})F_{k-1}F_k + F_{n-1}F_{k-1}^2)$$
$$= F_{k-1}^n + (-1)^n F_k^{n-2}((-1)^{k-1}F_{n-1} - (F_n - F_{n-1})F_{k-1}F_k + F_{n-1}F_{k-1}^2)$$

$$= F_{k-1}^n + (-1)^n F_n F_k^{n-1} F_{k-1} + (-1)^{n-1} F_{n-1} F_k^{n-2} (F_k^2 - F_{k-1}F_{k+1} + F_k F_{k-1} + F_{k-1}^2)$$
$$= F_{k-1}^n + (-1)^n F_n F_k^{n-1} F_{k-1} + (-1)^{n-1} F_{n-1} F_k^n.$$

Hence, (11) holds for n. According to the induction principle, (11) holds for any number $n \geq 2$. Thus, Lemma 3 is true.

Corollary 4: $g_{n,n} = F_{n-1}^n$.

Proof: Let $k = n$ in (11).

Lemma 4:

$$h_{n,k} = (-1)^{n-1} F_{n+k} F_k^{n-1}. \tag{13}$$

Proof: We obtain this by expanding the n^{th} row of the right side of (9).

Lemma 5:

$$g_{n,k} + h_{n,k} = (-1)^{n-1} L_n F_k^n + F_{k-1}^n. \tag{14}$$

Proof: By (11) and (13), and noticing that $F_{n+k} = F_{n+1}F_k + F_n F_{k-1}$, we have

$$\begin{aligned} g_{n,k} + h_{n,k} &= F_{k-1}^n + (-1)^{n-1} F_{n-1} F_k^n + (-1)^n F_n F_k^{n-1} F_{k-1} + (-1)^{n-1} (F_{n+1}F_k + F_n F_{k-1}) F_k^{n-1} \\ &= F_{k-1}^n + (-1)^{n-1} F_{n-1} F_k^n + (-1)^{n-1} F_{n+1} F_k^n \\ &= F_{k-1}^n + (-1)^{n-1} (F_{n-1} + F_{n+1}) F_k^n = F_{k-1}^n + (-1)^{n-1} L_n F_k^n. \end{aligned}$$

Hence, Lemma 5 holds.

Corollary 5:

$$\det(R_{n,n}) = \begin{vmatrix} F_{3n-1} & L_n F_{2n} & \cdots & L_n F_{3n-2} \\ F_{3n-2} & F_{3n-1} & \cdots & L_n F_{3n-3} \\ \cdots & \cdots & \cdots & \cdots \\ F_{2n} & F_{2n+1} & \cdots & F_{3n-1} \end{vmatrix} = F_{n-1}^n + (-1)^{n-1} F_{2n} F_n^{n-1}.$$

Proof: Let $k = n$ in Theorem 4 and note that $F_{2n} = L_n F_n$.

Remark: We can verify that our lemmas and Theorem 4 are also true for negative k.

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