PENTAGONAL NUMBERS IN THE PELL SEQUENCE AND DIOPHANTINE EQUATIONS $2x^2 = y^2(3y-1)^2 \pm 2$

V. Siva Rama Prasad and B. Srinivasa Rao
Department of Mathematics, Osmania University, Hyderabad - 500 007 A.P., India
(Submitted March 2000-Final Revision August 2000)

1. INTRODUCTION

It is well known that a positive integer N is called a *pentagonal* (*generalized pentagonal*) *number* if $N = m(3m-1)/2$ for some integer $m > 0$ (for any integer m).

Ming Luo [1] has proved that 1 and 5 are the only pentagonal numbers in the *Fibonacci sequence* $\{F_n\}$. Later, he showed (in [2]) that 2, 1, and 7 are the only generalized pentagonal numbers in the *Lucas sequence* $\{L_n\}$. In [3] we have proved that 1 and 7 are the only generalized pentagonal numbers in the *associated Pell sequence* $\{Q_n\}$ defined by

$$Q_0 = Q_1 = 1 \quad \text{and} \quad Q_{n+2} = 2Q_{n+1} + Q_n \quad \text{for } n \geq 0. \tag{1}$$

In this paper, we consider the *Pell sequence* $\{P_n\}$ defined by

$$P_0 = 0, P_1 = 1, \text{ and } P_{n+2} = 2P_{n+1} + P_n \quad \text{for } n \geq 0 \tag{2}$$

and prove that $P_{\pm 1}$, $P_{\pm 3}$, P_4, and P_6 are the only pentagonal numbers. Also we show that P_0, $P_{\pm 1}$, P_2, $P_{\pm 3}$, P_4, and P_6 are the only generalized pentagonal numbers. Further, we use this to solve the Diophantine equations of the title.

2. PRELIMINARY RESULTS

We have the following well-known properties of $\{P_n\}$ and $\{Q_n\}$: for all integers m and n,

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \text{ and } Q_n = \frac{\alpha^n + \beta^n}{2}, \text{ where } \alpha = 1 + \sqrt{2} \text{ and } \beta = 1 - \sqrt{2}, \tag{3}$$

$$P_{-n} = (-1)^{n+1} P_n \text{ and } Q_{-n} = (-1)^n Q_n, \tag{4}$$

$$Q_n^2 = 2P_n^2 + (-1)^n, \tag{5}$$

$$Q_{3n} = Q_n(Q_n^2 + 6P_n^2), \tag{6}$$

$$P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n}. \tag{7}$$

If m is odd, then:

$$\left.\begin{array}{l} \text{(i) } Q_m^2 + 6P_m^2 \equiv 7 \pmod 8, \quad \text{(ii) } P_m \equiv 1 \pmod 4, \\ \text{(iii) } Q_m \equiv \pm 1 \pmod 4 \text{ according as } m \equiv \pm 1 \pmod 4. \end{array}\right\} \tag{8}$$

Lemma 1: If n, k, and t are integers, then $P_{n+2kt} \equiv (-1)^{t(k+1)} P_n \pmod{Q_k}$.

Proof: If $t = 0$, the lemma is trivial and it can be proved for $t > 0$ by using induction on t with (7). If $t < 0$, say $t = -m$, where $m > 0$, then by (4) we have

$$P_{n+2kt} = P_{n-2km} = P_{n+2(-k)m} \equiv (-1)^{t(-k+1)} P_n \pmod{Q_{-k}} \equiv (-1)^{t(k+1)} P_n \pmod{Q_k},$$

proving the lemma.

3. SOME LEMMAS

Since $N = m(3m-1)/2$ if and only if $24N+1=(6m-1)^2$, we have that N is generalized pentagonal if and only if $24N+1$ is the square of an integer congruent to 5 (mod 6). Therefore, in this section we identify those n for which $24P_n+1$ is a perfect square.

We begin with

Lemma 2: Suppose $n \equiv \pm 1 \pmod{2^2 \cdot 5}$. Then $24P_n+1$ is a perfect square if and only if $n = \pm 1$.

Proof: If $n=\pm 1$, then by (4) we have $24P_n+1 = 24P_{\pm 1}+1 = 5^2$. Conversely, suppose $n = \pm 1$ $(\text{mod } 2^2 \cdot 5)$ and $n \notin \{-1, 1\}$. Then n can be written as $n = 2\cdot 11^r \cdot 5m \pm 1$, where $r \ge 0$, $11 \nmid m$, and $2 \mid m$. Taking

$$k = \begin{cases} 5m & \text{if } m \equiv \pm 2 \text{ or } \pm 8 \pmod{22}, \\ m & \text{otherwise,} \end{cases}$$

we get that

$$k \equiv \pm 4, \pm 6, \text{ or } \pm 10 \pmod{22}, \text{ and } n = 2kg \pm 1, \text{ where } g \text{ is odd (in fact, } g = 11^r \cdot 5 \text{ or } 11^r). \tag{9}$$

Now, by Lemma 1, (9), and (4), we get

$$\begin{aligned} 24P_n + 1 = 24P_{2kg\pm 1} + 1 &\equiv 24(-1)^{g(k+1)} P_{\pm 1} + 1 \pmod{Q_k} \\ &\equiv 24(-1)+1 \pmod{Q_k} \equiv -23 \pmod{Q_k}. \end{aligned}$$

Therefore, the Jacobi symbol

$$\left(\frac{24P_n+1}{Q_k}\right) = \left(\frac{-23}{Q_k}\right) = \left(\frac{Q_k}{23}\right). \tag{10}$$

But modulo 23, the sequence $\{Q_n\}$ has period 22. That is, $Q_{n+22t} \equiv Q_n \pmod{23}$ for all integers $t \ge 0$. Thus, by (9) and (4), we get $Q_k \equiv Q_{\pm 4}$, $Q_{\pm 6}$, or $Q_{\pm 10} \pmod{23} \equiv 17$, 7, or 5 (mod 23), so that

$$\left(\frac{Q_k}{23}\right) = \left(\frac{17}{23}\right), \left(\frac{7}{23}\right), \text{ or } \left(\frac{5}{23}\right),$$

and in any case

$$\left(\frac{Q_k}{23}\right) = -1. \tag{11}$$

From (10) and (11), it follows that

$$\left(\frac{24P_n+1}{Q_k}\right) = -1 \text{ for } n \notin \{-1, 1\},$$

showing $24P_n+1$ is not a perfect square. Hence, the lemma.

Lemma 3: Suppose $n \equiv \pm 3 \pmod{2^4}$. Then $24P_n+1$ is a perfect square if and only if $n = \pm 3$.

Proof: If $n = \pm 3$, then by (4) we have $24P_n+1 = 24P_{\pm 3}+1 = 11^2$. Conversely, suppose $n = \pm 3 \pmod{2^4}$ and $n \notin \{-3, 3\}$. Then n can be written as $n = 2\cdot 3^r \cdot k \pm 3$, where $r \ge 0$, $3 \nmid k$, and $8 \mid k$. And we get that

$$k \equiv \pm 8 \text{ or } \pm 16 \pmod{48} \text{ and } n = 2kg \pm 3, \text{ where } g = 3^r \text{ is odd and } k \text{ is even.} \tag{12}$$

Now, by Lemma 1, (12), and (4), we get

$$24P_n+1=24P_{2kg\pm 3}+1=24(-1)^{g(k+1)}P_{\pm 3}+1 \pmod{Q_k}\equiv -119 \pmod{Q_k}.$$

Hence, the Jacobi symbol

$$\left(\frac{24P_n+1}{Q_k}\right)=\left(\frac{-119}{Q_k}\right)=\left(\frac{Q_k}{119}\right). \tag{13}$$

But, modulo 119, the sequence $\{Q_n\}$ has period 48. Therefore, by (12) and (4), we get $Q_k \equiv Q_{\pm 8}$ or $Q_{\pm 16}$ (mod 119) $\equiv 101$ or 52 (mod 119), and in any case,

$$\left(\frac{Q_k}{119}\right)=-1. \tag{14}$$

From (13) and (14), it follows that

$$\left(\frac{24P_n+1}{Q_k}\right)=-1 \text{ for } n\notin\{-3,3\},$$

showing that $24P_n+1$ is not a perfect square. Hence the lemma.

Lemma 4: Suppose $n\equiv 4 \pmod{2^2\cdot 5}$. Then $24P_n+1$ is a perfect square if and only if $n=4$.

Proof: If $n=4$, then $24P_n+1=24P_4+1=17^2$. Conversely, suppose $n\equiv 4$ $(\text{mod } 2^2\cdot 5)$ and $n\neq 4$. Then n can be written as $n=2\cdot 3^r\cdot 5m+4$, where $r\geq 0$, $2|m$, and $3\nmid m$. Taking

$$k=\begin{cases} m & \text{if } m\equiv \pm 10 \pmod{30},\\ 5m & \text{otherwise,}\end{cases}$$

we get that

$$k\equiv \pm 10 \pmod{30} \text{ and } n=2kg+4, \text{ where } g \text{ is odd (in fact, } g=3^r \text{ or } 3^r\cdot 5). \tag{15}$$

Now, by Lemma 1 and (15), we get

$$24P_n+1=24P_{2kg+4}+1\equiv 24(-1)^{g(k+1)}P_4+1 \pmod{Q_k}\equiv -287 \pmod{Q_k}.$$

Hence, the Jacobi symbol

$$\left(\frac{24P_n+1}{Q_k}\right)=\left(\frac{-287}{Q_k}\right)=\left(\frac{Q_k}{287}\right). \tag{16}$$

But, modulo 287, the sequence $\{Q_n\}$ has period 30. Therefore, by (15) and (4), we get $Q_k\equiv Q_{\pm 10}$ (mod 287) $\equiv 206$ (mod 287), so that

$$\left(\frac{Q_k}{287}\right)=\left(\frac{206}{287}\right)=-1. \tag{17}$$

From (16) and (17), it follows that

$$\left(\frac{24P_n+1}{Q_k}\right)=-1 \text{ for } n\neq 4,$$

showing that $24P_n+1$ is not a perfect square. Hence the lemma.

Lemma 5: Suppose $n \equiv 2 \pmod{2^2 \cdot 5 \cdot 7}$. Then $24P_n + 1$ is a perfect square if and only if $n = 2$.

Proof: If $n = 2$, then we have $24P_n + 1 = 24P_2 + 1 = 7^2$. Conversely, suppose $n \equiv 2$ (mod $2^2 \cdot 5 \cdot 7$) and $n \neq 2$. Then n can be written as $n = 2 \cdot 23^r \cdot 5 \cdot 7m + 2$, where $r \geq 0$, $23 \nmid m$, and $2 \mid m$. Taking

$$k = \begin{cases} 7m & \text{if } m \equiv \pm 16 \pmod{46}, \\ 5m & \text{if } m = \pm 2, \pm 4, \pm 12, \pm 22 \pmod{46}, \\ m & \text{otherwise}, \end{cases}$$

we get that

$$k \equiv \pm 6, \pm 8, \pm 10, \pm 14, \pm 18, \pm 20 \pmod{46} \text{ and } n = 2kg + 2, \text{ where } g \text{ is odd (in fact, } g = 23^r \cdot 5 \cdot 7,\ 23^r \cdot 7, \text{ or } 23^r \cdot 5). \tag{18}$$

Now, by Lemma 1 and (18), we get

$$24P_n + 1 = 24P_{2kg+2} + 1 \equiv 24(-1)^{g(k+1)} P_2 + 1 \pmod{Q_k} \equiv -47 \pmod{Q_k}.$$

Hence, the Jacobi symbol

$$\left(\frac{24P_n + 1}{Q_k}\right) = \left(\frac{-47}{Q_k}\right) = \left(\frac{Q_k}{47}\right). \tag{19}$$

But, modulo 47, the sequence $\{Q_n\}$ has period 46. Therefore, by (18) and (4), we get $Q_k \equiv Q_{\pm 6}$, $Q_{\pm 8}$, $Q_{\pm 10}$, $Q_{\pm 14}$, $Q_{\pm 18}$, or $Q_{\pm 20}$ (mod 47) = 5, 13, 26, 33, 15, or 35 (mod 47), so that

$$\left(\frac{Q_k}{47}\right) = -1. \tag{20}$$

From (19) and (20), it follows that

$$\left(\frac{24P_n + 1}{Q_k}\right) = -1 \ \text{ for } n \neq 2,$$

showing $24P_n + 1$ is not a perfect square. Hence the lemma.

Lemma 6: Suppose $n \equiv 6 \pmod{2^2 \cdot 3 \cdot 5 \cdot 7}$. Then $24P_n + 1$ is a perfect square if and only if $n = 6$.

Proof: If $n = 6$, then we have $24P_n + 1 = 24P_6 + 1 = 41^2$. Conversely, suppose $n \equiv 6$ (mod $2^2 \cdot 3 \cdot 5 \cdot 7$) and $n \neq 6$. Then n can be written as $n = 2 \cdot 3^r \cdot 3 \cdot 5 \cdot 7m + 2$, where $r \geq 0$, $2 \mid m$, and $3 \nmid m$, which implies that $m \equiv \pm 2 \pmod 6$. Taking

$$k = \begin{cases} 3 \cdot 5m & \text{if } m \equiv \pm 2, \pm 32, \pm 52, \pm 76, \pm 82, \pm 86, \pm 100, \pm 124, \\ & \qquad \pm 130, \pm 170, \pm 178, \text{or } \pm 188 \pmod{396}, \\ 7m & \text{if } m \equiv \pm 26, \pm 62, \text{ or } \pm 88 \pmod{396}, \\ 3m & \text{if } m \equiv \pm 4, \pm 10, \pm 14, \pm 20, \pm 22, \pm 28, \pm 40, \pm 58, \pm 74, \pm 98, \pm 104, \\ & \qquad \pm 110, \pm 116, \pm 136, \pm 146, \pm 148, \pm 172, \text{ or } \pm 196 \pmod{396}, \\ m & \text{otherwise}, \end{cases}$$

we get that

$$\begin{aligned} k \equiv\ & \pm 8, \pm 12, \pm 16, \pm 30, \pm 34, \pm 38, \pm 42, \pm 44, \pm 46, \pm 48, \pm 50, \pm 56, \pm 60, \\ & \pm 64, \pm 66, \pm 68, \pm 70, \pm 80, \pm 84, \pm 92, \pm 94, \pm 102, \pm 106, \pm 112, \pm 118, \\ & \pm 120, \pm 122, \pm 128, \pm 134, \pm 140, \pm 142, \pm 152, \pm 154, \pm 158, \pm 160, \pm 164, \\ & \pm 166, \pm 174, \pm 176, \pm 182, \pm 184, \pm 190, \pm 192, \pm 194, \pm 202, \pm 204, \pm 206, \\ & \pm 212, \pm 214, \pm 220, \pm 222, \pm 230, \pm 232, \pm 236, \pm 238, \pm 242, \pm 244, \pm 254, \\ & \pm 256, \pm 262, \pm 268, \pm 274, \pm 276, \pm 278, \pm 284, \pm 290, \pm 294, \pm 302, \pm 304, \\ & \pm 312, \pm 316, \pm 326, \pm 328, \pm 330, \pm 332, \pm 336, \pm 340, \pm 346, \pm 348, \pm 350, \\ & \pm 352, \pm 354, \pm 358, \pm 362, \pm 366, \pm 380, \pm 384, \text{ or } \pm 388 \pmod{792} \end{aligned} \tag{21}$$

and

$$n = 2kg + 6, \text{ where } g \text{ is odd and } k \text{ is even.} \tag{22}$$

Now, by Lemma 1 and (22), we get

$$24P_n + 1 = 24P_{2kg+6} + 1 \equiv 24(-1)^{g(k+1)}P_6 + 1 \pmod{Q_k} \equiv -1679 \pmod{Q_k}.$$

Hence, the Jacobi symbol

$$\left(\frac{24P_n+1}{Q_k}\right) = \left(\frac{-1679}{Q_k}\right) = \left(\frac{Q_k}{1679}\right). \tag{23}$$

But, modulo 1679, the sequence $\{Q_n\}$ has period 792. Therefore, by (21) and (4), we get

$$\begin{aligned} Q_k \equiv\ & 577, 1132, 973, 485, 143, 1019, 923, 737, 141, 109, 513, 97, 329, 1015, \\ & 829, 601, 1098, 577, 1351, 1144, 513, 485, 362, 348, 1382, 1569, 1316, \\ & 316, 808, 163, 879, 1015, 1611, 1604, 973, 925, 1316, 923, 1151, 1019, \\ & 1589, 1382, 766, 1535, 1604, 329, 370, 163, 76, 1404, 26, 1385, 97, 122, \\ & 1535, 944, 1613, 143, 1589, 141, 1144, 1385, 1132, 370, 601, 1098, 1267, \\ & 582, 316, 109, 1175, 362, 348, 47, 1613, 766, 925, 582, 1351, 808, 139, 26, \\ & 76, 879, 1267, 122, 1569, \text{ or } 1175 \pmod{1679}, \text{ respectively.} \end{aligned}$$

And for all these values of k, the Jacobi symbol

$$\left(\frac{Q_k}{1679}\right) = -1. \tag{24}$$

From (23) and (24), it follows that

$$\left(\frac{24P_n+1}{Q_k}\right) = -1 \text{ for } n \neq 6,$$

showing that $24P_n + 1$ is not a perfect square. Hence the lemma.

Lemma 7: Suppose $n \equiv 0 \pmod{2 \cdot 3 \cdot 7^2 \cdot 13}$. Then $24P_n + 1$ is a perfect square if and only if $n = 0$.

Proof: If $n = 0$, then we have $24P_n + 1 = 24P_0 + 1 = 1^2$. Conversely, suppose $n \equiv 0$ (mod $2 \cdot 3 \cdot 7^2 \cdot 13$) and for $n \neq 0$ put $n = 2 \cdot 7^2 \cdot 13 \cdot 3^r \cdot z$, where $r \geq 1$ and $3 \nmid z$. We choose m as follows:

$$m=\begin{cases}13\cdot 3^r & \text{if } r\equiv\pm1 \pmod 4 \text{ according as } z\equiv\pm1 \pmod 3,\\ 7\cdot 3^r & \text{if } r\equiv\pm3 \pmod 4 \text{ according as } z\equiv\pm1 \pmod 3,\\ 7^2\cdot 3^r & \text{if } r\equiv 0 \pmod 4, z\equiv 1 \pmod 3 \text{ or } r\equiv 2 \pmod 4, z\equiv 2 \pmod 3,\\ 3^r & \text{if } r\equiv 2 \pmod 4, z\equiv 1 \pmod 3 \text{ or } r\equiv 0 \pmod 4, z\equiv 2 \pmod 3.\end{cases}$$

Then $n=2m(3k\pm1)$ for some integer k and odd m. Since, for $r\geq 1$, we have $3^r\equiv 3, 9, 27$, or 21 (mod 30) according as $r=1, 2, 3$, or 0 (mod 4), it follows that

$$m\equiv\pm9 \pmod{30} \text{ according as } z\equiv\pm1 \pmod 3. \tag{25}$$

Therefore, by Lemma 1, (4), (6), and the fact that m is odd, we have

$$\begin{aligned}24P_n+1=24P_{2(3m)k\pm2m}&\equiv 24(-1)^{k(3m+1)}P_{\pm2m}+1 \pmod{Q_{3m}}\\ &\equiv\pm24P_{2m}+1 \pmod{Q_m^2+6P_m^2} \text{ according as } z\equiv\pm1 \pmod 3.\end{aligned}$$

Letting $w_m=Q_m^2+6P_m^2$ and using (5), (7), and (8), we obtain the Jacobi symbol:

$$\begin{aligned}\left(\frac{24P_n+1}{w_m}\right)&=\left(\frac{\pm24P_{2m}+1}{w_m}\right)=\left(\frac{\pm48Q_mP_m-Q_m^2+2P_m^2}{w_m}\right)=\left(\frac{\pm48Q_mP_m+8P_m^2}{w_m}\right)\\ &=\left(\frac{2}{w_m}\right)\left(\frac{P_m}{w_m}\right)\left(\frac{\pm6Q_m+P_m}{w_m}\right)=\left(\frac{\pm6Q_m+P_m}{w_m}\right)=-\left(\frac{w_m}{\pm6Q_m+P_m}\right)\\ &=-\left(\frac{(\pm6Q_m+P_m)(\pm6Q_m-P_m)+217P_m^2}{\pm6Q_m+P_m}\right)=-\left(\frac{217}{\pm6Q_m+P_m}\right)\\ &=-\left(\frac{6Q_m\pm P_m}{217}\right)=-\left(\frac{H_m}{217}\right), \text{ where } H_m=6Q_m\pm P_m.\end{aligned}$$

But since

$$\text{modulo 217, the sequence } \{H_m\} \text{ is periodic with period 30.} \tag{26}$$

That is, $H_{n+30u}\equiv H_n \pmod{217}$ for all integers $u\geq 0$. And $H_{\pm9}=6Q_{\pm9}\pm P_{\pm9}\equiv\pm12 \pmod{217}$. Therefore, by (25) and (26), we get

$$\left(\frac{24P_n+1}{w_n}\right)=-\left(\frac{\pm12}{217}\right)=-1.$$

As a consequence of Lemmas 2-7, we have the following lemmas.

Lemma 8: Suppose $n\equiv 0, \pm1, 2, \pm3, 4$, or 6 (mod 152880). Then $24P_n+1$ is a perfect square if and only if $n\equiv 0, \pm1, 2, \pm3, 4$, or 6.

Lemma 9: $24P_n+1$ is not a perfect square if $n\not\equiv 0, \pm1, 2, \pm3, 4$, or 6 (mod 152880).

Proof: We prove the lemma in different steps, eliminating at each stage certain integers n congruent modulo 152880 for which $24P_n+1$ is not a square. In each step, we choose an integer m such that the period k (of the sequence $\{P_n\}$ mod m) is a divisor of 152880 and thereby eliminate certain residue classes modulo k. For example:

(a) Mod 41. The sequence $\{P_n\}$ mod 41 has period 10. We can eliminate $n\equiv 8 \pmod{10}$, since $24P_8+1\equiv 35 \pmod{41}$ and 35 is a quadratic nonresidue modulo 41. There remain $n\equiv 0$, 1, 2, 3, 4, 5, 6, 7, and 9 (mod 10) or, equivalently $n \equiv 0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17$, and 19 (mod 20).

(b) Mod 29. The sequence $\{P_n\}$ mod 29 has period 20. We can eliminate $n \equiv 7, 12, 13, 14, 16$, and 18 (mod 20), since they imply, respectively, $24P_n + 1 \equiv 26, 11, 26, 3, 3$, and 11 (mod 29). There remain $n \equiv 0, 1, 2, 3, 4, 5, 6, 9, 10, 11, 15, 17$, or 19 (mod 20) or, equivalently, $n \equiv 0, 1, 2, 3, 4, 5, 6, 9, 10, 11, 15, 17, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 35, 37$, or 39 (mod 40).

Similarly, we can eliminate the remaining values of n. After reaching modulo 152880, if there remain any values of n, we eliminate them in the higher modulos (i.e., in the multiples of 152880). We tabulate these in Tables A and B.

4. MAIN THEOREM

Theorem 1:

(a) P_n is a generalized pentagonal number only for $n = 0, \pm 1, 2, \pm 3, 4$, or 6.

(b) P_n is a pentagonal number only for $n = \pm 1, \pm 3, 4$, or 6.

Proof:

(a) From Lemmas 8 and 9, the first part of the theorem follows.

(b) Since an integer N is pentagonal if and only if $24N + 1 = (6m-1)^2$, where m is a positive integer, and since $P_0 = 0$, $P_2 = 2$, we have $24P_0 + 1 \neq (6m-1)^2$ and $24P_2 + 1 \neq (6m-1)^2$ for positive integer m, from which it follows that P_0 and P_2 are not pentagonal.

5. SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS

If D is a positive integer that is not a perfect square, it is well known that $x^2 - Dy^2 = \pm 1$ is called the Pell equation and that if $x_1 + y_1\sqrt{D}$ is the fundamental solution of it (i.e., x_1 and y_1 are least positive integers), then $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$ is also a solution of the same equation; conversely, every solution of it is of this form.

Now, by (5), we have $Q_n^2 = 2P_n^2 + (-1)^n$ for every n. Therefore, it follows that

$$Q_{2n} + \sqrt{2}P_{2n} \text{ is a solution of } x^2 - 2y^2 = 1, \tag{27}$$

while

$$Q_{2n+1} + \sqrt{2}P_{2n+1} \text{ is a solution of } x^2 - 2y^2 = -1. \tag{28}$$

Thus, the complete set of solutions of the equations $x^2 - 2y^2 = \pm 1$ is given by

$$x = \pm Q_n,\ y = \pm P_n. \tag{29}$$

Theorem 2: The solution set of the Diophantine equation

$$2x^2 = y^2(3y-1)^2 - 2 \tag{30}$$

is $\{(\pm 1, 1), (\pm 7, 2)\}$.

Proof: Writing $Y = y(3y-1)/2$, equation (30) reduces to the form

$$x^2 - 2Y^2 = -1, \tag{31}$$

whose solutions are, by (28), $Q_{2n+1} + \sqrt{2}P_{2n+1}$ for any integer n.

TABLE A

Modulus m	Period k	Required values of n where $\left(\frac{24P_n+1}{m}\right) = -1$	Left out values of n (mod t) where t is a positive integer
41	10	8.	0, ±1, 2, ±3, 4, 5 or 6 (mod 10)
29	20	7, 12, 13, 14 and 16.	0, ±1, 2, ±3, 4, ±5, 6, ± 9 or 10(mod 20)
19	40	5, 15, 17, 19, 21, 22, 23, 25, 26 and 35.	0, ±1, 2, ±3, 4, 6, ± 9, ±10, ±11 or 20 (mod 40)
59	40	24.	
241	80	±9, ±10, ±29, 30, ±31, ±39, 44 and 50.	0, ±1, 2, ±3, 4, 6, ±11, ±20, ±37, 40, 42 or 46 (mod 80)
31	30	±7, ±11, 12, 14, 24 and 26.	0, ±1, 2, ±3, 4, 6, ±60, 100, ±117, 120 or 122 (mod 240)
269	60	±9, ±17, ±21 and 22.	
601	120	46.	
2281	120	20 and 40.	
1153	48	±5, 8, 28, 30 and 32.	
239	14	±5, 7, 8 and 10.	0, ±1, 2, ±3, 4, 6, 420, 840 or 1260 (mod 1680)
13	28	±11, 16, 20 and 26.	
113	56	±25, ±27, 30, 40 and 46.	
337	56	12 and 18.	
71	70	60 and 62.	
83	168	28, ±69 and ±71.	
139	280	42.	
281	280	126.	
37633	336	±165 and 170.	
79	26	±7, 10, 13, 14, 20 and 22.	0, ±1, 2, ±3, 4, 6, 5460, 10920 or 16380 (mod 21840).
599	26	8, ±9, 16 and 24.	
313	78	±11, 18, ±25, ±27, 28, ±29, ±31, 32, ±37, 38, 58 and 64.	
521	260	±21, ±23, 44, 80, ±83, 160, 186, 240 and 246.	
1949	260	±37, ±57, ±63, ±81, 82 and 122.	
1091	312	52, 54 and 168.	
181	364	168, 286 and 338.	
1471	98	±11, 14, ±15, 16, ±17, 18, ±27, 28, ±29, 30, ±39, 46, 48, 56, 58, 60 and 76.	0, ±1, 2, ±3, 4, 6, 38220, 76440 or 114660 (mod 152880).
293	196	±25, ±31, ±53, ±55, 84, ±85, 86, 88, 140 and 172.	
587	1176	±335, 338, 510, 678, 756, 846, 1012 and 1014.	
2939	5880	2520 and 2522.	

We now eliminate: $n \equiv 38220, 76440,$ or $114660 \pmod{152880}$.
Or equivalently: $n \equiv 38220, 76440, 114660, 191100, 229320,$ or $267540 \pmod{305760}$.

TABLE B

Modulus m	Period k	Required values of n where $\left(\frac{24P_n+1}{m}\right) = -1$	Left out values of n (mod t) where t is a positive integer
97	96	±12, 36 and 60.	±76440 (mod 305760) or equivalently ±76440, ±229320 (mod 611520)
449	448	56, 168.	Completely eliminated under modulo 611520.
2689	1344	840, 1176.	

Now $x=a$, $y=b$ is a solution of (30) $\Leftrightarrow a+\sqrt{2}b(3b-1)/2$ is a solution of (31) $\Leftrightarrow a=Q_{2n+1}$ and $b(3b-1)/2=P_{2n+1}$ for some integer n. But we know by Theorem 1(a) that P_k is generalized pentagonal if and only if $k=0, \pm1, 2, \pm3, 4$, or 6. Therefore, we have either

(i) $a=Q_{-1}=-1,\ b(3b-1)/2=P_{-1}=1;$ (ii) $a=Q_1=1,\ b(3b-1)/2=P_1=1;$

(iii) $a=Q_{-3}=-7,\ b(3b-1)/2=P_{-3}=5;$ (iv) $a=Q_3=7,\ b(3b-1)/2=P_3=5.$

Solving the above equations, we get the required solution set of equation (30).

We can prove the following theorem in a similar manner.

Theorem 3: The solution set of the Diophantine equation $2x^2=y^2(3y-1)^2+2$ is

$$\{(\pm1,0),(\pm3,-1),(\pm17,3),(\pm99,-280)\}.$$

REFERENCES

1. Ming Luo. "Pentagonal Numbers in the Fibonacci Sequence." In *Fibonacci Numbers and Their Applications* **6**. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1995.
2. Ming Luo. "Pentagonal Numbers in the Lucas Sequence." *Portugaliae Mathematica* **53.3** (1996):325-29.
3. V. Siva Rama Prasad & B. Srinivasa Rao. "Pentagonal Numbers in the Associated Pell Sequence and Diophantine Equations $x^2(3x-1)^2=8y^2\pm4$." To appear in *The Fibonacci Quarterly*.

AMS Classification Numbers: 11B39, 11D25, 11B37

❖❖❖