AN OLYMPIAD PROBLEM, EULER'S SEQUENCE, AND STIRLING'S FORMULA

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1. INTRODUCTION

There are several ways of defining the real number e. The most common of them is to define e as the limit of the nondecreasing sequence

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n\geq 1}.$$

Related to this definition is the following problem proposed in 1990 at the Romanian County Olympiad: "Study the convergence of the sequence $\{x_n\}_{n\geq 1}$ defined by

$$\left(1+\frac{1}{n}\right)^{n+x_n}=e."$$

The problem is not hard to solve, but, surprisingly, a different approach to solving it than the one given originally by the proposers yields some interesting applications. The solution given by the proposers used l'Hôpital's rule. For this, we write

$$x_n=\frac{1}{\ln\left(1+\frac{1}{n}\right)}-n,$$

and then obtain

$$\lim_{x\to\infty}\frac{1-x\ln\left(1+\frac{1}{x}\right)}{\ln\left(1+\frac{1}{x}\right)}=\frac{1}{2}.$$

If one were to solve the problem in a different way, then a natural question related to convergence would be whether the sequence is bounded or not. The answer to this is given by the double inequality

$$\left(1+\frac{1}{n}\right)^n<e<\left(1+\frac{1}{n}\right)^{n+1}, \tag{1}$$

which proves that the sequence is bounded and $x_n\in(0,1)$. In view of this, one might ask if (1) can be refined to a similar pair of inequalities that incorporate 0.5 in the exponents. In other words, is sit true that, for a given $\varepsilon>0$ and n sufficiently large, the following inequalities hold:

$$\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}-\varepsilon}<e<\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}+\varepsilon}?$$

In order to answer this question, we will generalize (1) and show how the *generalized α-inequality* can be applied to various problems, namely: find a shorter proof of Stirling's formula than the one given by D. S. Mitrinovic (see [3], pp. 181-84), solve the Olympiad problem mentioned before, and study the convergence of a general Euler-type series.

2. THE α-INEQUALITY

We prove the following

Proposition:

(a) If $0 \le \alpha < 0.5$, then there exists an $x(\alpha) \ge 0$ such that

$$\left(1+\frac{1}{x}\right)^{x+\alpha} < e, \quad x > x(\alpha);$$

(b) If $\alpha \ge 0.5$, then

$$\left(1+\frac{1}{x}\right)^{x+\alpha} > e, \quad x > 0.$$

Proof: For $\alpha \ge 0$, let $f_\alpha : (0, \infty) \to (0, \infty)$, $f_\alpha(x) = \left(1+\frac{1}{x}\right)^{x+\alpha}$. Logarithmic differentiation of this function yields

$$f_\alpha'(x) = \left(1+\frac{1}{x}\right)^{x+\alpha}\left[\ln\left(1+\frac{1}{x}\right) - \frac{\alpha+x}{x(x+1)}\right].$$

If we consider now the mapping $g_\alpha : (0, \infty) \to \mathbf{R}$, $g_\alpha(x) = \ln\left(1+\frac{1}{x}\right) - \frac{\alpha+x}{x(x+1)}$, then

$$g_\alpha'(x) = \frac{-x(1-2\alpha)+\alpha}{x^2(x+1)^2}.$$

We notice a couple of cases:

(i) If $\alpha \in [0, 0.5)$, then $g_\alpha'(x) < 0$ for all $x > x(\alpha) = \alpha/(1-2\alpha)$. Thus, g_α is nonincreasing on $(x(\alpha), \infty)$ and $g_\alpha(x) > \lim_{x\to\infty} g_\alpha(x) = 0$ for all $x > x(\alpha)$. This implies that $f_\alpha'(x) > 0$ for all $x > x(\alpha)$. Hence, f_α is strictly increasing on $(x(\alpha), \infty)$. Finally, using the fact that $\lim_{x\to\infty} f_\alpha(x) = e$, we infer that $f_\alpha(x) < e$ for all $x > x(\alpha)$.

(ii) If $\alpha \in [0.5, \infty)$, then $g_\alpha'(x) > 0$ for all $x > 0$. From this point, an argument similar to the one used before leads to the conclusion that $f_\alpha(x) > e$ for all $x > 0$.

Before we continue with our applications, let us note that the case $\alpha = 0.5$ is treated, among other inequalities involving exponentials, in [3, §3.6].

3. APPLICATIONS

A. If we let $\varepsilon \in (0, 0.5)$ and $\alpha = 0.5 - \varepsilon$ in (a), we see that $x_n > \frac{1}{2} - \varepsilon$ for all $n \ge [(1-2\varepsilon]/4\varepsilon] + 1$. By (b), it is true that $x_n < \frac{1}{2} + \varepsilon$ for any $n \ge 1$; hence,

$$\left|x_n - \frac{1}{2}\right| < \varepsilon, \quad n \ge n(\varepsilon) = \left[\frac{1-2\varepsilon}{4\varepsilon}\right] + 1,$$

which proves that the sequence converges, indeed, to 0.5.

B. It is well known that Euler's sequence

$$\gamma_n = 1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\ln n,\ n\ge 2,$$

is nondecreasing and converges to Euler's constant, C = .57721566.... We show below that this fact is just a complex consequence of the α-inequality with $\alpha = 0, 1$ in the previous section. More generally, we use our proposition to study the convergence of the family of sequences:

$$\gamma_n(a)=\frac{1}{1+a}+\frac{1}{2+a}+\cdots+\frac{1}{n-1+a}-\ln n,\ n\ge 2, a\ge 0.$$

We will prove that $(\gamma_n(a))$ is convergent for any $a\ge 0$. Since

$$\gamma_{n+1}(a)-\gamma_n(a)=\frac{1}{n+a}-\ln\left(1+\frac{1}{n}\right),$$

if $a<0.5$, then $\gamma_{n+1}(a)-\gamma_n(a)>0$ for all $n\ge n(a)=\left[\frac{a}{1-2a}\right]+2$, and if $a\ge 0.5$, then $\gamma_{n+1}(a)-\gamma_n(a)<0$ for all $n\ge 2$. If we could prove that our sequence is also bounded, then convergence would follow automatically. Let us consider first the case when $a\in[0,0.5)$. Since $a+1\ge 1$, we can write $\gamma_{k+1}(a+1)-\gamma_k(a+1)<0$, $k\ge 2$. But

$$\gamma_k(a+1)=\gamma_{k+1}(a)+\ln\left(1+\frac{1}{k}\right)-\frac{1}{a+1}$$

implies that

$$\ln\left(1+\frac{1}{k}\right)-\ln\left(1+\frac{1}{k+1}\right)>\gamma_{k+2}(a)-\gamma_{k+1}(a),\ k\ge 2.$$

Now, if we let $k=2,3,\ldots,n-2$ and add these inequalities, we find that

$$\gamma_n(a)<\ln\frac{3}{2}+\gamma_3(a)-\ln\left(1+\frac{1}{n-1}\right)<\frac{1}{1+a}+\frac{1}{2+a}-\ln 2,\ n\ge 4,$$

which proves that our sequence is bounded and, hence, convergent. Denote its limit by $\gamma(a)$. Note that $\gamma(a)\in[m(a), M(a)]$, where $m(a)=\min\{\gamma_2(a),\ldots,\gamma_{n(a)}(a)\}$ and

$$M(a)=\max\left\{\gamma_2(a),\gamma_3(a),\frac{1}{1+a}+\frac{1}{2+a}-\ln 2\right\}.$$

For $a=0$, $\gamma(0)=C$ and $n(0)=2$; hence, $C\in[1-\ln 2, 1.5-\ln 2]$. Suppose now that $a=0.5$. An easy computation gives

$$\gamma_n\left(\frac{1}{2}\right)=\gamma_{2n}-\gamma_n+2\ln 2-2\to C+2\ln 2-2.$$

When $a\in(0.5,1]$, we have

$$\gamma_n(a)\ge\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n=\gamma_{n+1}+\ln\left(1+\frac{1}{n}\right)-1$$

which implies $\gamma_n(a)\to\gamma(a)\in\left[C-1,\frac{1}{1+a}-\ln 2\right]$. Finally, if $a\in(1,\infty)$, then

$$\begin{aligned}\gamma_n-\gamma_n(a)&=a\left(\frac{1}{1+a}+\frac{1}{2(2+a)}+\cdots+\frac{1}{(n-1)(n-1+a)}\right)\\&<a\left(\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n-1}-\frac{1}{n}\right)=\frac{a}{4}-\frac{a}{n};\end{aligned}$$

hence, $\gamma_n(a) \to \gamma(a) \in \left[C - \frac{a}{4}, \frac{1}{1+a} - \ln 2\right]$.

C. Stirling's formula asserts that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

It is well known that the result is closely related to the behavior of the gamma function, $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ for large values of x. This classical way of deriving Stirling's formula can be found, for example, in [1, pp. 20-24]. For different approaches, see also [2] and [4]. We use our proposition to give a proof which is different from the ones mentioned before. This proof uses an argument similar to, but shorter than, the one given by D. S. Mitrinovic. We will assume as known the following result due to Wallis:

$$\lim_{n\to\infty} \frac{(2n)!!}{(2n-1)!!\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}.$$

For $\alpha > 0$, let $u_n(\alpha) = \frac{n!}{n^{n+\alpha}e^{-n}}$, $n \geq 2$. Then

$$\frac{u_n\left(\frac{1}{2}\right)}{u_{n+1}\left(\frac{1}{2}\right)} = \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} > 1;$$

thus, $\left(u_n\left(\frac{1}{2}\right)\right)$ is nonincreasing and bounded below by 1. Therefore, $\lim_{n\to\infty} u_n\left(\frac{1}{2}\right) = u$ exists and is strictly positive. Note also that

$$\frac{u_n^2\left(\frac{1}{2}\right)}{u_{2n}\left(\frac{1}{2}\right)} = \frac{(2n)!!\sqrt{2}}{(2n-1)!!\sqrt{n}}.$$

If we let $n \to \infty$, we obtain $u = \sqrt{2\pi}$, which proves Stirling's formula. Note that in this formula the value $\alpha = 0.5$ is the best one, for

$$\lim_{n\to\infty} u_n(\alpha) = \begin{cases} \infty & \text{if } \alpha < 0.5, \\ 0 & \text{if } \alpha > 0.5. \end{cases}$$

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