ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2003. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-940 ***Proposed by Gabriela Stănică & Pantelimon Stănică, Auburn Univ. Montgomery, Montgomery, AL***

How many perfect squares are in the sequence

$$x_n = 1 + \sum_{k=0}^{n} F_k! \text{ for } n \geq 0?$$

B-941 ***Proposed by Walther Janous, Innsbruck, Austria***

Show that

$$\frac{nF_{n+6}}{2^{n+1}} + \frac{F_{n+8}}{2^n} - F_8 < 0.$$

B-942 ***Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA***

(a) For $n > 3$, find the Fibonacci number closest to L_n.

(b) For $n > 3$, find the Fibonacci number closest to L_n^2.

B-943 ***Proposed by José Luis Diaz & Juan J. Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain***

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \frac{L_k^2}{F_k} \geq \frac{(L_{n+2}-3)^2}{F_{n+2}-1}.$$

When does equality occur?

B-944 ***Proposed by Paul S. Bruckman, Berkeley, CA***

For all odd primes p, prove that

$$L_p \equiv 1 - \frac{p}{2} \sum_{k=1}^{p-1} \frac{L_k}{k} \pmod{p^2}$$

where $\frac{1}{k}$ represents the residue $k^{-1} \pmod{p}$.

B-945 ***Proposed by N. Gauthier, Royal Military College of Canada***

For $n \geq 0$, $q > 0$, s integers, show that

$$\sum_{l=0}^{n} \binom{n}{l} F_{q-1}^{l} F_{(q+1)(n-l)+s} = F_{q+1}^{n} F_{2n+s}.$$

SOLUTIONS

Some Sum Divides Another

B-925 ***Proposed by José Luis Diaz & Juan J. Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain***
(Vol. 39, no. 5, November 2001)

Prove that $\sum_{k=0}^{n} F_{k+1}^2$ divides

$$\sum_{k=0}^{n} F_{k+1}^2 [F_{k+2} + (-1)^k F_k] \quad \text{for } n \geq 0.$$

Solution by H.-J. Seiffert, Berlin, Germany

From (I_3) of [1], we know that

$$\sum_{k=0}^{n} F_{k+1}^2 = F_{n+1} F_{n+2},$$

so it suffices to prove that, for all $n \geq 0$,

$$S_n := \sum_{k=0}^{n} F_{k+1}^2 [F_{k+2} + (-1)^k F_k] = \left(\frac{1-(-1)^n}{2} F_{n+3} + (-1)^n F_{n+2} \right) F_{n+1} F_{n+2}. \tag{1}$$

Direct computation shows that this is true for $n = 0$. Assuming that (1) holds for $n-1$, $n \geq 1$, we obtain

$$\begin{aligned} S_n &= S_{n-1} + F_{n+1}^2 [F_{n+2} + (-1)^n F_n] \\ &= \left(\frac{1+(-1)^n}{2} F_{n+2} - (-1)^n F_{n+1} \right) F_n F_{n+1} + F_{n+1}^2 [F_{n+2} + (-1)^n F_n] \\ &= \left(\frac{1+(-1)^n}{2} F_n + F_{n+1} \right) F_{n+1} F_{n+2} = \left(\frac{1-(-1)^n}{2} F_{n+3} + (-1)^n F_{n+2} \right) F_{n+1} F_{n+2}, \end{aligned}$$

where the latter equality is easily established by considering the cases in which n is even and n is odd. This completes the induction proof of (1).

Reference

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Almost all solvers used essentially a similar method and provided the same reference.

Also solved by Brian D. Beasley, Paul S. Bruckman, Charles Cook, Kenneth B. Davenport, L. A. G. Dresel, Russell J. Hendel, Walther Janous, and the proposers.

Find the Limit

B-926 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan (Vol. 39, no. 5, November 2001)***

If $1 < a < \alpha$, evaluate

$$\lim_{n\to\infty}\left(a^{\frac{1}{F_1}+\frac{1}{F_2}+\cdots+\frac{1}{F_n}} - a^{\frac{1}{F_1}+\frac{1}{F_2}+\cdots+\frac{1}{F_{n-1}}}\right).$$

Solution by Gurdial Arora & Vlazko Kocic, New Orleans

To find the above limit, we use the following result from [1]:

$$\lim_{n\to\infty}\left(\frac{1}{F_1}+\frac{1}{F_2}+\cdots+\frac{1}{F_n}\right) = 2+\alpha \approx 3.6180339\ldots.$$

Therefore,

$$\lim_{n\to\infty}\left(a^{\frac{1}{F_1}+\frac{1}{F_2}+\cdots+\frac{1}{F_n}} - a^{\frac{1}{F_1}+\frac{1}{F_2}+\cdots+\frac{1}{F_{n-1}}}\right) = 0.$$

Reference

1. Zdzislaw W. Trzaska. "On Factorial Fibonacci Numbers." *The Math. Gazette* (1998):82-85.

Almost all solvers noted that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{F_k}$$

exists and gave several references. In fact, the result is not difficult to prove.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Russell J. Hendel, Walther Janous, H.-J. Seiffert, Naim Tugler, and the proposer.

A More General Identity

B-927 ***Proposed by R. S. Melham, University of Technology, Sydney, Australia (Vol. 39, no. 5, November 2001)***

G. Candido ["A Relationship between the Fourth Powers of the Terms of the Fibonacci Series," *Scripta Mathematica* **17.3-4** (1951):230] gave the following fourth-power relation:

$$2(F_n^4+F_{n+1}^4+F_{n+2}^4) = (F_n^2+F_{n+1}^2+F_{n+2}^2)^2.$$

Generalize this relation to the sequence defined for all integers n by

$$W_n = pW_{n-1} - qW_{n-2},\ \ W_0 = a,\ W_1 = b.$$

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

We find $2(q^4W_n^4 + p^4W_{n+1} + W_{n+2}^4) = (q^2W_n^2 + p^2W_{n+1}^2 + W_{n+2}^2)^2$, because

$$(q^2W_n^2 + p^2W_{n+1}^2 + W_{n+2}^2)^2 = q^4W_n^4 + p^4W_{n+1}^4 + W_{n+2}^4 + q^2W_n^2(p^2W_{n+1}^2 + W_{n+2}^2) + p^2W_{n+1}^2(q^2W_n^2 + W_{n+2}^2) + W_{n+2}^2(q^2W_n^2 + p^2W_{n+1}^2),$$

and we complete the proof by noting that

$$\begin{aligned} q^2W_n^2(p^2W_{n+1}^2 + W_{n+2}^2) &= q^2W_n^2[(W_{n+2} - pW_{n+1})^2 + 2pW_{n+1}W_{n+2}] \\ &= q^4W_n^4 + 2pq^2W_n^2W_{n+1}W_{n+2}, \\ p^2W_{n+1}^2(q^2W_n^2 + W_{n+2}^2) &= p^2W_{n+1}^2[(W_{n+2} + qW_n)^2 - 2qW_nW_{n+2}] \\ &= p^4W_{n+1}^4 - 2p^2qW_nW_{n+1}^2W_{n+2}, \\ W_{n+2}^2(q^2W_n^2 + p^2W_{n+1}^2) &= W_{n+2}^2[(pW_{n+1} - qW_n)^2 + 2pqW_nW_{n+1}] \\ &= W_{n+2}^4 + 2pqW_nW_{n+1}W_{n+2}^2, \end{aligned}$$

and

$$\begin{aligned} &2pq^2W_n^2W_{n+1}W_{n+2} - 2p^2qW_nW_{n+1}^2W_{n+2} + 2pqW_nW_{n+1}W_{n+2}^2 \\ &= 2pqW_nW_{n+1}W_{n+2}(qW_n - pW_{n+1} + W_{n+2}) = 0. \end{aligned}$$

Also solved by Brian D. Beasley, Paul S. Bruckman, L. A. G. Dresel, Russell J. Hendel, Walther Janous, and the proposer.

A Complex Fibonacci Polynomial

B-928 *Proposed by H.-J. Seiffert, Berlin, Germany*
(Vol. 39, no. 5, November 2001)

The Fibonacci polynomials are defined by $F_0(x) = 0$, $F_1(x) = 1$, $F_{n+1}(x) = xF_{n+1}(x) + F_n(x)$ for $n \ge 0$. Show that, for all complex numbers x and all nonnegative integers n,

$$F_{2n+1}(x) = \sum_{k=0}^{n} (-1)^{\lceil k/2 \rceil} \binom{n - \lceil k/2 \rceil}{\lfloor k/2 \rfloor} (x^2 + 2)^{n-k},$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor- and ceiling-function, respectively.

Solution by Paul S. Bruckman, Berkeley, CA

We may restate Seiffert's putative identity as follows:

$$F_{2n+1}(x) = G_{2n+1}(x), \tag{1}$$

where

$$G_{2n+1}(x) = \sum_{k=0}^{n} (-1)^{[(k+1)/2]} \binom{n - [(k+1)/2]}{[k/2]} (x^2 + 2)^{n-k}. \tag{2}$$

Our proof of (1) uses a modified form of induction. First, however, we derive the recurrence satisfied by the $F_{2n+1}(x)$'s. Note that the basic recurrence satisfied by the $F_n(x)$'s has the characteristic equation, $z^2 - xz - 1 = 0$, which has the solutions $u = u(x) = x + \theta$ and $v = v(x) = x - \theta$, where $\theta = (x^2 + 4)^{1/2}$. Note that $u + v = x$ and $uv = -1$. Therefore, the characteristic equation of the $F_{2n+1}(x)$'s is as follows: $(z - u^2)(z - v^2) = 0$, i.e., $z^2 - (x^2 + 2)z + 1 = 0$. In other words, the $F_{2n+1}(x)$'s satisfy the following recurrence:

$$F_{2n+5}(x) = (x^2+2)F_{2n+3}(x) - F_{2n+1}(x), \; n = 0, 1, \ldots. \tag{3}$$

Note that $F_1(x) = 1 = G_1(x)$. Also, $F_3(x) = x^2 + 1 = (x^2+2) - 1 = G_3(x)$. Now

$$(x^2+2)G_{2n+3}(x) - G_{2n+1}(x) = \sum_{k=0}^{n+1} (-1)^{[(k+1)/2]} \binom{n+1-[(k+1)/2]}{[k/2]} (x^2+2)^{n+2-k}$$
$$-\sum_{k=0}^{n} (-1)^{[(k+1)/2]} \binom{n-[(k+1)/2]}{[k/2]} (x^2+2)^{n-k}.$$

In the last sum, we replace k by $k-2$; thus, this sum becomes

$$+\sum_{k=2}^{n+2} (-1)^{[(k+1)/2]} \binom{n+1-[(k+1)/2]}{[k/2]-1} (x^2+2)^{n+2-k}.$$

We may extend this last sum by including the terms for $k = 0$ and $k = 1$, since the combinatorial term vanishes for such values. Similarly, the first sum may be extended to include the term for $k = n+2$, for the same reason. We also note that

$$\binom{n+1-[(k+1)/2]}{[k/2]} + \binom{n+1-[(k+1)/2]}{[k/2]-1} = \binom{n+2-[(k+1)/2]}{[k/2]}.$$

Accordingly, we obtain the following result:

$$(x^2+2)G_{2n+3}(x) - G_{2n+1}(x) = \sum_{k=0}^{n+2} (-1)^{[(k+1)/2]} \binom{n+2-[(k+1)/2]}{[k/2]} (x^2+2)^{n+2-k},$$

which we recognize to equal $G_{2n+5}(x)$. Therefore, the $F_{2n+1}(x)$'s and the $G_{2n+1}(x)$'s satisfy the same recurrence, and also have the same initial values. It follows that

$$F_{2n+1}(x) = G_{2n+1}(x), \; n = 0, 1, \ldots, \text{ for all } x. \text{ Q.E.D.}$$

Also solved by Walther Janous and the proposer.

Between Fibonacci, Lucas, and Legendre

B-929 ***Proposed by Harvey J. Hindin, Huntington Station, NY (Vol. 39, no. 5, November 2001)***

Prove that

A) $$F_{2N} = (1/5^{1/2}) \sum_{K=0}^{2N-1} P_K(5^{1/2}/2) P_{2N-1-K}(5^{1/2}/2) \text{ for } N \ge 1$$

and

B) $$L_{2N+1} = \sum_{K=0}^{2N} P_K(5^{1/2}/2) P_{2N-K}(5^{1/2}/2) \text{ for } N \ge 0,$$

where $P_K(x)$ is the Legendre polynomial given by $P_0(x) = 1$, $P_1(x) = x$, and the recurrence relation $(K+1)P_{K+1}(x) = (2K+1)xP_K(x) - KP_{K-1}(x)$.

Solution by H.-J. Seiffert, Berlin, Germany

The sequences of Fibonacci and Lucas polynomials are defined by

$$F_0(x) = 0, \; F_1(x) = 1, \text{ and } F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \; n \ge 0,$$

and

$$L_0(x)=2,\ L_1(x)=x, \text{ and } L_{n+2}(x)=xL_{n+1}(x)+L_n(x),\ n\ge 0,$$

respectively. We shall prove that, for all real numbers x,

A) $F_{2N}(x)=\dfrac{x}{\sqrt{x^2+4}}\sum_{K=0}^{2N-1}P_K(\sqrt{x^2+4}/2)P_{2N-1-K}(\sqrt{x^2+4}/2)$ for $N\ge 1$,

and

B) $L_{2N+1}(x)=x\sum_{K=0}^{2N}P_K(\sqrt{x^2+4}/2)P_{2N-K}(\sqrt{x^2+4}/2)$ for $N\ge 0$.

It is known from equations (1.7) and (1.8) of [1] that

$$F_n(x)=\frac{\alpha(x)^n-\beta(x)^n}{\sqrt{x^2+4}} \text{ and } L_n(x)=\alpha(x)^n+\beta(x)^n,\ n\ge 0, \tag{1}$$

where $\alpha(x)=(x+\sqrt{x^2+4})/2$ and $\beta(x)=(x-\sqrt{x^2+4})/2$. For sufficiently small $|z|$, let

$$G(z)=\sum_{L=1}^{\infty}(\alpha(x)^L-(-1)^L\beta(x)^L)z^{L-1}.$$

Then, by (1),

$$G(z)=\sqrt{x^2+4}\sum_{N=1}^{\infty}F_{2N}(x)z^{2N-1}+\sum_{N=0}^{\infty}L_{2N+1}(x)z^{2N}. \tag{2}$$

On the other hand, the known closed form expression for infinite geometric sums gives

$$G(z)=\frac{\alpha(x)}{1-\alpha(x)z}+\frac{\beta(x)}{1+\beta(x)z},$$

so that, by $\alpha(x)+\beta(x)=x$, $\beta(x)=-1$, $\alpha(x)\beta(x)=-1$, and $\alpha(x)-\beta(x)=\sqrt{x^2+4}$,

$$G(z)=\frac{x}{1-\sqrt{x^2+4}z+z^2}. \tag{3}$$

The Legendre polynomials have the generating function (see [2], p. 190)

$$\sum_{L=0}^{\infty}P_L(x)z^L=\frac{1}{\sqrt{1-2xz+z^2}} \text{ for small } |z|.$$

Squaring, replacing x by $\sqrt{x^2+4}/2$, and multiplying the obtained identity by x, in view of (3) we get

$$\sum_{L=0}^{\infty}\left(x\sum_{K=0}^{L}P_K(\sqrt{x^2+4}/2)P_{L-K}(\sqrt{x^2+4}/2)\right)z^L=G(z). \tag{4}$$

The above stated identities A) and B) now follow from (2) and (4) by comparing coefficients.

Taking $x=1$ solves the present proposal.

Remarks: By analytic continuation, the identities A) and B) remain valid for all complex numbers x. Other identities involving Fibonacci and Lucas numbers can be obtained by taking $x=\sqrt{5}$, 4, $1/\sqrt{5}$, $3i$, etc. For example, since $F_{2N}(\sqrt{5})=\sqrt{5}F_{4N}/3$, from A) with $x=\sqrt{5}$, we find

$$F_{4N} = \sum_{K=0}^{2N-1} P_K(3/2)P_{2N-1-K}(3/2) \text{ for } N \geq 1.$$

References

1. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.
2. F. G. Tricomi. *Vorlesungen über Orthogonalreihen*. 2. Auflage, Springer, 1970.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Ovidiu Furdui, Walther Janous, and the proposer.

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Author and Title Index

The TITLE, AUTHOR, ELEMENTARY PROBLEMS, ADVANCED PROBLEMS, and KEY-WORD indices for Volumes 1-38.3 are now on The Fibonacci Web Page. Anyone wanting their own copies may request them from **Charlie Cook** at **The University of South Carolina, Sumter**, by **e-mail at <ccook@sc.edu>**. Copies will be sent by e-mail attachment. PLEASE INDICATE WORDPERFECT 6.1, MS WORD 97, or WORDPERFECT DOS 5.1.