FORMULAS FOR CONVOLUTION FIBONACCI NUMBERS AND POLYNOMIALS

Guodong Liu
Dept. of Mathematics, Huizhou University, Huizhou, Guangdong, 516015, People's Republic of China
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1. INTRODUCTION

The Fibonacci numbers F_n $(n=0,1,2,\dots)$ satisfy the recurrence relation $F_n=F_{n-1}+F_{n-2}$ $(n\geq 2)$ with $F_0=0$, $F_1=1$. We denote

$$F(n,k)=\sum_{v_1+v_2+\cdots+v_k=n} F_{v_1}F_{v_2}\dots F_{v_k} \quad (n\geq k), \tag{1}$$

where the summation is over all k-dimension nonnegative integer coordinates $(v_1,v_2,\dots,v_k)$ such that $v_1+v_2+\cdots+v_k=n$ and k is any positive integer. The numbers $F(n,k)$ are called *convolution Fibonacci numbers* (see [3], [1], [2]). W. Zhang recently studied the convolution Fibonacci numbers $F(n,2)$, $F(n,3)$, and $F(n,4)$ in [4], and the following three identities were obtained:

$$\sum_{a+b=n} F_aF_b=\frac{1}{5}((n-1)F_n+2nF_{n-1}), \tag{2}$$

$$\sum_{a+b+c=n} F_aF_bF_c=\frac{1}{50}((5n^2-9n-2)F_{n-1}+(5n^2-3n-2)F_{n-2}), \tag{3}$$

$$\sum_{a+b+c+d=n} F_aF_bF_cF_d=\frac{1}{150}((4n^3-12n^2-4n+12)F_{n-2}+(3n^3-6n^2-3n+6)F_{n-3}). \tag{4}$$

The main purpose of this paper is that a recurrence relation and an expression in terms of Fibonacci numbers are given for convolution Fibonacci numbers $F(n,k)$, where n and k are any positive integers with $n\geq k$.

2. DEFINITIONS AND LEMMAS

Definition 1: The k^{th}-order Fibonacci numbers $F_n^{(k)}$ are given by the following expansion formula:

$$\left(\frac{t}{1-t-t^2}\right)^k=\sum_{n=0}^{\infty}F_n^{(k)}t^n. \tag{5}$$

By (1) and (5), we have $F_n^{(1)}=F_n$, $F(n,k)=F_n^{(k)}$, and $F_n^{(k)}=0$ $(n<k)$.

Definition 2: The k^{th}-order Fibonacci polynomials $F_n^{(k)}(x;p)$ are given by the following expansion formula:

$$\left(\frac{1}{1-2xt-pt^2}\right)^k=\sum_{n=0}^{\infty}F_n^{(k)}(x;p)t^n. \tag{6}$$

By (5) and (6), we have $F_n^{(k)}=F_{n-k}^{(k)}(\frac{1}{2};1)$ $(n\geq k)$.

Definition 3: Let n, k, j be three integers with $n \geq k \geq 2$, $0 \leq j \leq k-1$, and

$$M_{k-1-j,j} = \left\{ (x_1, x_2, \dots, x_{k-1}) \mid x_i = 0 \text{ or } 1 \; (i = 1, 2, \dots, k-1) \text{ and } \sum_{i=1}^{k-1} x_i = k-1-j \right\}.$$

For any $(x_1, x_2, \dots, x_{k-1}) \in M_{k-1-j,j}$, $\lambda(x_1, x_2, \dots, x_{k-1}; k, n)$ is defined by

$$\lambda_{k-1-j,j}(x_1, x_2, \dots, x_{k-1}; k, n) = \left(\frac{y_1}{k-1} + z_1\right)\left(\frac{y_2}{k-2} + z_2\right) \cdots \left(\frac{y_{k-1}}{1} + z_{k-1}\right),$$

where $y_1, y_2, \dots, y_{k-1}, z_1, z_2, \dots, z_{k-1}$ satisfies the following:

(a) If $x_1 = 1$, then $y_1 = n$; if $x_1 = 0$, then $y_1 = n-1$.

(b) $\forall i : 1 \leq i \leq k-1$; if $x_i = 1$, then $z_i = -1$; if $x_i = 0$, then $z_i = 1$.

(c) $\forall i : 1 \leq i \leq k-2$; if $x_i = x_{i+1} = 1$ or $x_i = 0$, $x_{i+1} = 1$, then $y_{i+1} = y_i$; if $x_i = x_{i+1} = 0$ or $x_i = 1$, $x_{i+1} = 0$, then $y_{i+1} = y_i - 1$.

Lemma 1:

(a) $$\frac{d}{dx} F_n^{(k)}(x; p) = 2k F_{n-1}^{(k+1)}(x; p) \quad (n \geq 1); \tag{7}$$

(b) $$(n+1) F_{n+1}^{(k)}(x; p) = 2x(n+k) F_n^{(k)}(x; p) + p(n+2k-1) F_{n-1}^{(k)}(x; p); \tag{8}$$

(c) $$\frac{d}{dx} F_{n+1}^{(k)}(x; p) - 2x \frac{d}{dx} F_n^{(k)}(x; p) - 2k F_n^{(k)}(x; p) - p \frac{d}{dx} F_{n-1}^{(k)}(x; p) = 0. \tag{9}$$

Proof: By Definition 2. □

Lemma 2: For $k \geq 2$, we have:

(a) $$x \frac{d}{dx} F_n^{(k)}(x; p) + p \frac{d}{dx} F_{n-1}^{(k)}(x; p) = n F_n^{(k)}(x; p); \tag{10}$$

(b) $$\frac{d}{dx} F_n^{(k)}(x; p) - x \frac{d}{dx} F_{n-1}^{(k)}(x; p) = (n-1+2k) F_{n-1}^{(k)}(x; p). \tag{11}$$

Proof: By Lemma 1(b) and (c), we immediately obtain (10) and (11). □

Lemma 3: We denote

$$s(n, k, j) := \sum_{(x_1, x_2, \dots, x_{k-1}) \in M_{k-1-j,j}} \lambda_{k-1-j,j}(x_1, x_2, \dots, x_{k-1}; k, n) \quad (0 \leq j \leq k-1),$$

where the summation is over all $(k-1)$-dimension coordinates $(x_1, x_2, \dots, x_{k-1})$ such that $(x_1, x_2, \dots, x_{k-1}) \in M_{k-1-j,j}$, then:

(a) $\left(\frac{n}{k} - 1\right) s(n, k, 0) = s(n, k+1, 0)$;

(b) $\left(\frac{n-1}{k} + 1\right) s(n-1, k, k-1) = s(n, k+1, k)$;

(c) $\left(\frac{n}{k} - 1\right) s(n, k, j) + \left(\frac{n-1}{k} + 1\right) s(n-1, k, j-1) = s(n, k+1, j) \quad (1 \leq j \leq k-1)$.

Proof:

(a) $$\left(\frac{n}{k}-1\right)s(n,k,0)=\left(\frac{n}{k}-1\right)\sum_{(x_1,x_2,\ldots,x_{k-1})\in M_{k-1,0}}\lambda_{k-1,0}(x_1,x_2,\ldots,x_{k-1};k,n)$$
$$=\left(\frac{n}{k}-1\right)\lambda_{k-1,0}(1,1,\ldots,1;k,n)=\left(\frac{n}{k}-1\right)\left(\frac{n}{k-1}-1\right)\left(\frac{n}{k-2}-1\right)\cdots\left(\frac{n}{1}-1\right)$$
$$=\lambda_{k,0}(1,1,\ldots,1;k+1,n)=\sum_{(x_1,x_2,\ldots,x_k)\in M_{k,0}}\lambda_{k,0}(x_1,x_2,\ldots,x_k;k+1,n)=s(n,k+1,0).$$

(b) $$\left(\frac{n-1}{k}+1\right)s(n-1,k,k-1)=\left(\frac{n-1}{k}+1\right)\sum_{(x_1,x_2,\ldots,x_{k-1})\in M_{0,k-1}}\lambda_{0,k-1}(x_1,x_2,\ldots,x_{k-1};k,n-1)$$
$$=\left(\frac{n-1}{k}+1\right)\lambda_{0,k-1}(0,0,\ldots,0;k,n-1)=\left(\frac{n-1}{k}+1\right)\left(\frac{n-2}{k-1}+1\right)\cdots\left(\frac{n-k}{1}+1\right)$$
$$=\lambda_{0,k}(0,0,\ldots,0;k+1,n)=\sum_{(x_1,x_2,\ldots,x_k)\in M_{0,k}}\lambda_{0,k}(x_1,x_2,\ldots,x_k;k+1,n)=s(n,k+1,k).$$

(c) $$s(n,k+1,j)=\sum_{(x_1,x_2,\ldots,x_k)\in M_{k-j,j}}\lambda_{k-j,j}(x_1,x_2,\ldots,x_k;k+1,n)$$
$$=\sum_{(1,x_2,\ldots,x_k)\in M_{k-j,j}}\lambda_{k-j,j}(1,x_2,\ldots,x_k;k+1,n)+\sum_{(0,x_2,\ldots,x_k)\in M_{k-j,j}}\lambda_{k-j,j}(0,x_2,\ldots,x_k;k+1,n)$$
$$=\left(\frac{n}{k}-1\right)\sum_{(x_2,\ldots,x_k)\in M_{k-1-j,j}}\lambda_{k-1-j,j}(x_2,\ldots,x_k;k,n)$$
$$+\left(\frac{n-1}{k}+1\right)\sum_{(x_2,\ldots,x_k)\in M_{k-j,j-1}}\lambda_{k-j,j-1}(x_2,\ldots,x_k;k,n-1)$$
$$=\left(\frac{n}{k}-1\right)s(n,k,j)+\left(\frac{n-1}{k}+1\right)s(n-1,k,j-1).\ \square$$

3. MAIN RESULTS

Theorem 1: For $n\ge k\ge 2$, we have:

(a) $$F_n^{(k)}(x;p)=\frac{x}{2(x^2+p)}\left(\frac{n+k}{k-1}-1\right)F_{n+1}^{(k-1)}(x;p)+\frac{p}{2(x^2+p)}\left(\frac{n+k-1}{k-1}+1\right)F_n^{(k-1)}(x;p);\tag{12}$$

(b) $$F_n^{(k)}=\frac{1}{5}\left(\frac{n}{k-1}-1\right)F_n^{(k-1)}+\frac{2}{5}\left(\frac{n-1}{k-1}+1\right)F_{n-1}^{(k-1)}.\tag{13}$$

Proof:

(a) By (10) and (11), we have

$$(x^2+p)\frac{d}{dx}F_n^{(k)}(x;p)=nxF_n^{(k)}(x;p)+p(n-1+2k)F_{n-1}^{(k)}(x;p).\tag{14}$$

By (14) and (7), we immediately obtain (12).

(b) Taking $x=\frac{1}{2}$ and $p=1$ in (12) and noting that

$$F_n^{(k)}=F_{n-k}^{(k)}\left(\frac{1}{2};1\right),$$

we immediately obtain (13). □

Theorem 2: For $n\geq k\geq 2$, we have

$$F_{n-k}^{(k)}(x;p)=\sum_{j=0}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-1-j}\left(\frac{p}{2(x^2+p)}\right)^{j}s(n,k,j)F_{n-1-j}(x;p), \tag{15}$$

where $s(n,k,j)$ is defined as in Lemma 3.

Proof (using mathematical induction):

1° When $k=2$, by Theorem 1, we have

$$\begin{aligned}F_{n-2}^{(2)}(x;p)&=\frac{x}{2(x^2+p)}(n-1)F_{n-1}(x;p)+\frac{p}{2(x^2+p)}nF_{n-2}(x;p)\\&=\frac{x}{2(x^2+p)}\lambda_{1,0}(1;2,n)F_{n-1}(x;p)+\frac{p}{2(x^2+p)}\lambda_{0,1}(0;2,n)F_{n-2}(x;p)\\&=\sum_{j=0}^{1}\left(\frac{x}{2(x^2+p)}\right)^{1-j}\left(\frac{p}{2(x^2+p)}\right)^{j}\sum_{(x_1)\in M_{1-j,j}}\lambda_{1-j,j}(x_1;2,n)F_{n-1-j}(x;p)\\&=\sum_{j=0}^{1}\left(\frac{x}{2(x^2+p)}\right)^{1-j}\left(\frac{p}{2(x^2+p)}\right)^{j}s(n,2,j)F_{n-1-j}(x;p).\end{aligned} \tag{16}$$

(16) shows that (15) is true for the natural number 2.

2° Suppose that (15) is true for some natural number k. By the supposition, Theorem 1, and Lemma 3, we have

$$F_{n-(k+1)}^{(k+1)}(x;p)=\frac{x}{2(x^2+p)}\left(\frac{n}{k}-1\right)F_{n-k}^{(k)}(x;p)+\frac{p}{2(x^2+p)}\left(\frac{n-1}{k}+1\right)F_{n-1-k}^{(k)}(x;p)$$

$$=\sum_{j=0}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^{j}\left(\frac{n}{k}-1\right)s(n,k,j)F_{n-1-j}(x;p)$$

$$+\sum_{j=0}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-1-j}\left(\frac{p}{2(x^2+p)}\right)^{j+1}\left(\frac{n-1}{k}+1\right)s(n-1,k,j)F_{n-2-j}(x;p)$$

$$=\sum_{j=0}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^{j}\left(\frac{n}{k}-1\right)s(n,k,j)F_{n-1-j}(x;p)$$

$$+\sum_{j=1}^{k}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^{j}\left(\frac{n-1}{k}+1\right)s(n-1,k,j-1)F_{n-1-j}(x;p)$$

$$=\left(\frac{x}{2(x^2+p)}\right)^{k}\left(\frac{n}{k}-1\right)s(n,k,0)F_{n-1}(x;p)+$$

$$+\sum_{j=1}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^{j}\left(\left(\frac{n}{k}-1\right)s(n,k,j)+\left(\frac{n-1}{k}+1\right)s(n-1,k,j-1)\right)F_{n-1-j}(x;p)$$

$$+\left(\frac{p}{2(x^2+p)}\right)^{k}\left(\frac{n-1}{k}+1\right)s(n-1,k,k-1)F_{n-1-k}(x;p)$$

$$=\left(\frac{x}{2(x^2+p)}\right)^{k}s(n,k+1,0)F_{n-1}(x;p)$$

$$+\sum_{j=1}^{k-1}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^{j}s(n,k+1,j)F_{n-1-j}(x;p)$$

$$+\left(\frac{p}{2(x^2+p)}\right)^{k}s(n,k+1,k)F_{n-1-k}(x;p)$$

$$=\sum_{j=0}^{k}\left(\frac{x}{2(x^2+p)}\right)^{k-j}\left(\frac{p}{2(x^2+p)}\right)^{j}s(n,k+1,j)F_{n-1-j}(x;p). \tag{17}$$

(17) shows that (15) is also true for the natural number $k+1$. □

From 1° and 2°, we know that (15) is true.

Theorem 3: For $n \ge k \ge 2$, we have

$$F(n,k)=F_n^{(k)}=\left(\frac{1}{5}\right)^{k-1}\sum_{j=0}^{k-1}2^j s(n,k,j)F_{n-j}, \tag{18}$$

where $s(n,k,j)$ is defined as in Lemma 3.

Proof: Taking $x=\frac{1}{2}$ and $p=1$ in Theorem 2, and noting that

$$F(n,k)=F_n^{(k)}=F_{n-k}^{(k)}\left(\frac{1}{2},1\right) \quad\text{and}\quad F_{n-j}=F_{n-1-j}\left(\frac{1}{2};1\right),$$

we immediately obtain (18). □

Corollary 1: For $n \ge k \ge 2$, we have

(a) $F(n,2)=\frac{1}{5}((n-1)F_n+2nF_{n-1})$;

(b) $F(n,3)=\frac{1}{50}((n^2-3n+2)F_n+(4n^2-6n-4)F_{n-1}+(4n^2-4)F_{n-2})$;

(c) $F(n,4)=\frac{1}{750}((n^3-6n^2+11n-6)F_n+(6n^3-24n^2+6n+36)F_{n-1}$
$+(12n^3-24n^2-48n+36)F_{n-2}+(8n^3-32n)F_{n-3})$;

(d) $F(n,5)=\frac{1}{15000}((n^4-10n^3+35n^2-50n+24)F_n+(8n^4-60n^3+100n^2+120n-288)F_{n-1}$
$+(24n^4-120n^3-60n^2+660n-144)F_{n-2}+(32n^4-80n^3-320n^2+440n+288)F_{n-3}$
$+(16n^4-160n^2+144)F_{n-4})$.

Remark: By Corollary 1(a)-(c) and $F_n=F_{n-1}+F_{n-2}$ $(n\ge 2)$, we immediately obtain (2), (3), and (4) (see Zhang [4]).

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