SOME IDENTITIES INVOLVING THE FIBONACCI POLYNOMIALS*

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1. INTRODUCTION AND RESULTS

As usual, the Fibonacci polynomials $F(x)=\{F_n(x)\}$, $n=0,1,2,\dots$, are defined by the second-order linear recurrence sequence

$$F_{n+2}(x)=xF_{n+1}(x)+F_n(x) \tag{1}$$

for $n\geq 0$ and $F_0(x)=0$, $F_1(x)=1$. Let

$$\alpha=\frac{x+\sqrt{x^2+4}}{2} \quad\text{and}\quad \beta=\frac{x-\sqrt{x^2+4}}{2}$$

denote the roots of the characteristic polynomial $\lambda^2-x\lambda-1$ of the sequence $F(x)$, then the terms of the sequence $F(x)$ (see [2]) can be expressed as

$$F_n(x)=\frac{1}{\alpha-\beta}\{\alpha^n-\beta^n\}$$

for $n=0,1,2,\dots$.

If $x=1$, then the sequence $F(1)$ is called the Fibonacci sequence, and we shall denote it by $F=\{F_n\}$.

The various properties of $\{F_n\}$ were investigated by many authors. For example, Duncan [1] and Kuipers [3] proved that $(\log F_n)$ is uniformly distributed mod 1. Robbins [4] studied the Fibonacci numbers of the forms $px^2\pm 1$ and $px^3\pm 1$, where p is a prime. The second author [5] obtained some identities involving the Fibonacci numbers. The main purpose of this paper is to study how to calculate the summation involving the Fibonacci polynomials:

$$\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1+1}(x)\cdot F_{a_2+1}(x)\cdot\cdots\cdot F_{a_k+1}(x), \tag{2}$$

where the summation is over all k-dimension nonnegative integer coordinates $(a_1,a_2,\dots,a_k)$ such that $a_1+a_2+\cdots+a_k=n$, and k is any positive integer.

Regarding (2), it seems that it has not been studied yet, at least I have not seen expressions like (2) before. The problem is interesting because it is a generalization of [5], and it can also help us to find some new convolution properties for $F(x)$. In this paper we use the generating function of the sequence $F(x)$ and its partial derivative to study the evaluation of (2), and give an interesting identity for any fixed positive integers k and n. That is, we shall prove the following proposition.

Proposition: Let $F(x)=\{F_n(x)\}$ be defined by (1). Then, for any positive integers k and n, we have the calculating formula

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$$\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1+1}(x)\cdot F_{a_2+1}(x)\cdot\cdots\cdot F_{a_k+1}(x)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n+k-1-m}{m}\cdot\binom{n+k-1-2m}{k-1}\cdot x^{n-2m},$$

where $\binom{m}{n}=\frac{m!}{n!(m-n)!}$, and $[z]$ denotes the greatest integer not exceeding z.

From this proposition, we may immediately deduce the following several corollaries.

Corollary 1: For any positive integers k and n, we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n+k} F_{a_1}\cdot F_{a_2}\cdot\cdots\cdot F_{a_k}=\sum_{m=0}^{[\frac{n}{2}]}\binom{n+k-1-m}{m}\cdot\binom{n+k-1-2m}{k-1}.$$

Corollary 2: For any positive integers k and n, we have

$$\sum_{a_1+\cdots+a_k=n+k} F_{2a_1}\cdot F_{2a_2}\cdot\cdots\cdot F_{2a_k}=3^k\cdot 5^{\frac{n-k}{2}}\cdot\sum_{m=0}^{[\frac{n}{2}]}\frac{\binom{n+k-1-m}{m}\cdot\binom{n+k-1-2m}{k-1}}{5^m}.$$

Corollary 3: The identity

$$\sum_{a_1+\cdots+a_k=n+k} F_{3a_1}\cdot F_{3a_2}\cdot\cdots\cdot F_{3a_k}=2^{2n+k}\cdot\sum_{m=0}^{[\frac{n}{2}]}\frac{\binom{n+k-1-m}{m}\cdot\binom{n+k-1-2m}{k-1}}{16^m}$$

holds for all positive integers k and n.

Corollary 4: Let k and n be positive integers. Then

$$\sum_{a_1+\cdots+a_k=n+k} F_{4a_1}\cdot F_{4a_2}\cdot\cdots\cdot F_{4a_k}=3^n\cdot 7^k\cdot 5^{\frac{n-k}{2}}\cdot\sum_{m=0}^{[\frac{n}{2}]}\frac{\binom{n+k-1-m}{m}\cdot\binom{n+k-1-2m}{k-1}}{45^m}.$$

Corollary 5: Let k and n be positive integers. Then

$$\sum_{a_1+\cdots+a_k=n+k} F_{5a_1}\cdot F_{5a_2}\cdot\cdots\cdot F_{5a_k}=5^k\cdot 11^n\cdot\sum_{m=0}^{[\frac{n}{2}]}\frac{\binom{n+k-1-m}{m}\cdot\binom{n+k-1-2m}{k-1}}{121^m}.$$

In fact, for any positive integer m, using the proposition, we can give an exact calculating formula for

$$\sum_{a_1+a_2+\cdots+a_k=n+k} F_{ma_1}\cdot F_{ma_2}\cdot\cdots\cdot F_{ma_k}.$$

2. PROOF OF THE PROPOSITION

In this section we shall complete the proof of the proposition. First, note that

$$F_n(x)=\frac{1}{\sqrt{x^2+4}}\left[\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n-\left(\frac{x-\sqrt{x^2+4}}{2}\right)^n\right],$$

so we can easily deduce that the generating function of $F(x)$ is

$$G(t,x)=\frac{1}{1-xt-t^2}=\frac{1}{(\alpha-t)(\beta-t)} \\ =\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^{n+1}-\beta^{n+1})\cdot t^n=\sum_{n=0}^{\infty}F_{n+1}(x)\cdot t^n. \tag{3}$$

Let $\frac{\partial G^k(t,x)}{\partial x^k}$ denote the k^{th} partial derivative of $G(t,x)$ for x, and $F_n^{(k)}(x)$ denote the k^{th} derivative of $F_n(x)$. Then from (3) we have

$$\frac{\partial G(t,x)}{\partial x}=\frac{t}{(1-xt-t^2)^2}=\sum_{n=0}^{\infty}F_{n+1}^{(1)}(x)\cdot t^n, \\ \frac{\partial G^2(t,x)}{\partial x^2}=\frac{2!\cdot t^2}{(1-xt-t^2)^3}=\sum_{n=0}^{\infty}F_{n+1}^{(2)}(x)\cdot t^n, \\ \cdots \\ \frac{\partial G^{k-1}(t,x)}{\partial x^{k-1}}=\frac{(k-1)!\cdot t^{k-1}}{(1-xt-t^2)^k}=\sum_{n=0}^{\infty}F_{n+1}^{(k-1)}(x)\cdot t^n=\sum_{n=0}^{\infty}F_{n+1}^{(k-1)}(x)\cdot t^{n+k-1}, \tag{5}$$

where we have used the fact that $F_{n+1}(x)$ is a polynomial of degree n.

For any two absolutely convergent power series $\sum_{n=0}^{\infty}a_nx^n$ and $\sum_{n=0}^{\infty}b_nx^n$, note that

$$\left(\sum_{n=0}^{\infty}a_nx^n\right)\cdot\left(\sum_{n=0}^{\infty}b_nx^n\right)=\sum_{n=0}^{\infty}\left(\sum_{u+v=n}a_ub_v\right)x^n.$$

So from (5) we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{a_1+\cdots+a_k=n}F_{a_1+1}(x)\cdot F_{a_2+1}(x)\cdot\cdots\cdot F_{a_k+1}(x)\right)\cdot t^n=\left(\sum_{n=0}^{\infty}F_{n+1}(x)\cdot t^n\right)^k \\ =\frac{1}{(1-xt-t^2)^k}=\frac{1}{(k-1)!\cdot t^{k-1}}\frac{\partial G^{k-1}(t,x)}{\partial x^{k-1}}=\frac{1}{(k-1)!}\sum_{n=0}^{\infty}F_{n+k}^{(k-1)}(x)\cdot t^n. \tag{6}$$

Equating the coefficients of t^n on both sides of equation (6), we obtain the identity

$$\sum_{a_1+a_2+\cdots+a_k=n}F_{a_1+1}(x)\cdot F_{a_2+1}(x)\cdot\cdots\cdot F_{a_k+1}(x)=\frac{1}{(k-1)!}\cdot F_{n+k}^{(k-1)}(x). \tag{7}$$

On the other hand, note that from the combinatorial identity

$$\binom{n-m+1}{m}=\binom{n-m}{m}+\binom{n-m}{m-1}, \tag{8}$$

the recurrence formula $F_{n+2}(x)=xF_{n+1}(x)+F_n(x)$, and by mathematical induction, we can easily deduce

$$F_{n+1}(x)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n-m}{m}\cdot x^{n-2m}. \tag{9}$$

In fact, from the definition of $F_n(x)$, we know that (9) is true for $n=0$ and $n=1$. Assume (9) is true for all integers $0\le n\le k$. Then, for $n=k+1$, applying (8) and the inductive hypothesis we immediately obtain

$$\begin{aligned}\sum_{m=0}^{\left[\frac{k+1}{2}\right]}\binom{k+1-m}{m}\cdot x^{k+1-2m} &= 1+\sum_{m=0}^{\left[\frac{k-1}{2}\right]}\binom{k-m}{m+1}\cdot x^{k-1-2m}\\ &= 1+\sum_{m=0}^{\left[\frac{k-1}{2}\right]}\binom{k-1-m}{m+1}\cdot x^{k-1-2m}+\sum_{m=0}^{\left[\frac{k-1}{2}\right]}\binom{k-1-m}{m}\cdot x^{k-1-2m}\\ &= \sum_{m=0}^{\left[\frac{k+1}{2}\right]}\binom{k-m}{m}\cdot x^{k+1-2m}+\sum_{m=0}^{\left[\frac{k-1}{2}\right]}\binom{k-1-m}{m}\cdot x^{k-1-2m}\\ &= x\sum_{m=0}^{\left[\frac{k}{2}\right]}\binom{k-m}{m}\cdot x^{k-2m}+\sum_{m=0}^{\left[\frac{k-1}{2}\right]}\binom{k-1-m}{m}\cdot x^{k-1-2m}\\ &= xF_{k+1}(x)+F_k(x)=F_{k+2}(x),\end{aligned}$$

where we have used $\binom{k-m}{m}=0$ if $m>\frac{k}{2}$. So by induction we know that (9) is true for all non-negative integer n.

From (9) we can deduce that the $(k-1)^{\text{th}}$ derivative of $F_{n+k}(x)$ is

$$F_{n+k}^{(k-1)}(x)=\left(\sum_{m=0}^{\left[\frac{n+k-1}{2}\right]}\binom{n+k-1-m}{m}\cdot x^{n+k-1-2m}\right)^{(k-1)}=\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(n+k-1-m)!}{m!\cdot(n-2m)!}x^{n-2m}. \tag{10}$$

Combining (7) and (10), we obtain the identity

$$\sum_{a_1+\cdots+a_k=n}F_{a_1+1}(x)\cdot F_{a_2+1}(x)\cdot\cdots\cdot F_{a_k+1}(x)=\sum_{m=0}^{\left[\frac{n}{2}\right]}\binom{n+k-1-m}{m}\cdot\binom{n+k-1-2m}{k-1}\cdot x^{n-2m}.$$

This completes the proof of the Proposition.

Proof of the Corollaries: Taking $x=1$ in the Proposition and noting that $F_0=0$, we have

$$\begin{aligned}\sum_{a_1+a_2+\cdots+a_k=n}F_{a_1+1}(1)\cdot F_{a_2+1}(1)\cdot\cdots\cdot F_{a_k+1}(1) &= \sum_{a_1+1+a_2+1+\cdots+a_k+1=n+k}F_{a_1+1}\cdot F_{a_2+1}\cdot\cdots\cdot F_{a_k+1}\\ =\sum_{a_1+a_2+\cdots+a_k=n+k}F_{a_1}\cdot F_{a_2}\cdot\cdots\cdot F_{a_k} &= \sum_{m=0}^{\left[\frac{n}{2}\right]}\binom{n+k-1-m}{m}\cdot\binom{n+k-1-2m}{k-1}.\end{aligned}$$

This proves Corollary 1.

Taking $x=-\sqrt{5}, 4, -3\sqrt{5}$, and 11, respectively, in the Proposition, and noting that

$$F_n(-\sqrt{5})=\frac{(-1)^{n+1}}{3}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n-\left(\frac{3-\sqrt{5}}{2}\right)^n\right]=\frac{(-1)^{n+1}\sqrt{5}}{3}\cdot F_{2n},$$

$$F_n(4)=\frac{1}{2\sqrt{5}}\left[(2+\sqrt{5})^n-(2-\sqrt{5})^n\right]=\frac{1}{2\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{3n}-\left(\frac{1-\sqrt{5}}{2}\right)^{3n}\right]=\frac{1}{2}\cdot F_{3n},$$

$$F_n(-3\sqrt{5})=\frac{(-1)^{n+1}}{7}\left[\left(\frac{7+3\sqrt{5}}{2}\right)^n-\left(\frac{7-3\sqrt{5}}{2}\right)^n\right]=\frac{(-1)^{n+1}\sqrt{5}}{7}\cdot F_{4n},$$

and

$$F_n(11) = \frac{1}{5\sqrt{5}}\left[\left(\frac{11+5\sqrt{5}}{2}\right)^n - \left(\frac{11-5\sqrt{5}}{2}\right)^n\right] = \frac{1}{5\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{5n} - \left(\frac{1-\sqrt{5}}{2}\right)^{5n}\right] = \frac{1}{5}\cdot F_{5n},$$

we may immediately deduce Corollary 2, Corollary 3, Corollary 4, and Corollary 5.

This completes the proof of the Corollaries.

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