EULERIAN POLYNOMIALS AND RELATED EXPLICIT FORMULAS

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1. INTRODUCTION

For $\alpha \in \mathbb{C} \setminus \{1\}$ we write, as in [2],

$$\frac{1-\alpha}{e^t-\alpha}=\sum_{n=0}^{\infty} H_n(\alpha)\frac{t^n}{n!},$$

from which it follows that $H_n(\alpha)$ $(n=0,1,\ldots)$ are uniquely determined by

$$H_0(\alpha)\equiv 1, \quad H_n(\alpha)=\frac{1}{\alpha-1}\sum_{k=0}^{n-1}\binom{n}{k}H_k(\alpha) \quad (n\geq 1). \tag{1}$$

The Eulerian polynomials $R_n(\alpha)$ $(n=0,1,\ldots)$ are defined by $R_n(\alpha)=(\alpha-1)^n H_n(\alpha)$ as Euler first discussed them in [4]. For $n \geq 1$, as is easily seen from (1), $R_n(\alpha)$ is a polynomial in α of degree $n-1$ with integer coefficients and was expressed by Euler in [4] as

$$R_n(\alpha)=\sum_{k=1}^{n} A_k^n \alpha^{k-1}, \tag{2}$$

where the integers A_k^n $(1\leq k\leq n)$ are known as Eulerian numbers (see also [3, p. 51]). Later, Frobenius [5] gave another expression for $R_n(\alpha)$ as

$$R_n(\alpha)=\sum_{k=1}^{n} k!\,S_k^n(\alpha-1)^{n-k}, \tag{3}$$

where S_k^n $(1\leq k\leq n)$ denote the Stirling numbers of the second kind (see also [3, p. 244]).

The object of this paper is to obtain one more expression for $R_n(\alpha)$ in terms of an array of integers C_k^n closely related to the central factorial numbers (see [6, §6.5]). By means of the new expression for $R_n(\alpha)$, we derive explicit formulas for Bernoulli and Euler numbers and others, and unify some known results, in terms of these C_k^n.

2. A NEW EXPRESSION FOR $R_n(\alpha)$

We define an array of integers C_k^n in the following way: for integers $r,\ k\geq 1$,

$$C_k^n=\begin{cases}\dfrac{1}{k}\displaystyle\sum_{j=1}^{k}(-1)^{k-j}\binom{2k}{k-j}j^{2r} & \text{if } n=2r-1,\\[2ex] kC_k^{2r-1} & \text{if } n=2r.\end{cases} \tag{4}$$

Clearly, $C_1^{2r-1}=C_1^{2r}=1$. We make the convention that $C_0^{2r-1}=C_0^{2r}=0$.

These integers C_k^n are closely related to the central factorial numbers of the second kind $T(n,k)$ defined as in [6, p. 212]. Indeed, for $r,k\geq 1$, $C_k^{2r-1}=(2k-1)!\,T(2r,2k)$. Thus, it follows from the properties of $T(2r,2k)$ (see also [1, pp. 428-29]) that

$$C_k^{2r-1}=\begin{cases}(2r-1)! & \text{if } k=r,\\ 0 & \text{if } k\geq r+1.\end{cases} \tag{5}$$

Moreover, the second formula in the definition (4) together with

$$C_k^{2r+1}=2(2k-1)C_{k-1}^{2r}+kC_k^{2r} \tag{6}$$

gives the recurrence for C_k^n. We may also derive (5) and (6) directly from the definition.

The new expression of $R_n(\alpha)$ given below contains the powers of α as in (2) and also that of $\alpha-1$ as in (3). Moreover, the number of the terms in the summation is about half of that in (2) and (3).

Theorem 1: For an integer $r\geq 1$:

$$R_{2r-1}(\alpha)=\sum_{k=1}^{r}C_k^{2r-1}\alpha^{k-1}(\alpha-1)^{2r-2k}, \tag{7}$$

$$R_{2r}(\alpha)=(1+\alpha)\sum_{k=1}^{r}C_k^{2r}\alpha^{k-1}(\alpha-1)^{2r-2k}. \tag{8}$$

Proof: Clearly, from (1), $R_1(\alpha)=1$ and $R_2(\alpha)=1+\alpha$. For the general case, the proof is by induction on $r\geq 1$ using the recurrence

$$R_{n+1}(\alpha)=(n+1)\alpha R_n(\alpha)+(1-\alpha)\frac{d}{d\alpha}(\alpha R_n(\alpha)) \tag{9}$$

for $n\geq 1$ (see [2], [5]). If (7) is true, then by (9),

$$\begin{aligned}R_{2r}(\alpha)&=(2r)\alpha R_{2r-1}(\alpha)+(1-\alpha)\frac{d}{d\alpha}(\alpha R_{2r-1}(\alpha))\\&=\sum_{k=1}^{r}kC_k^{2r-1}(1+\alpha)\alpha^{k-1}(\alpha-1)^{2r-2k},\end{aligned}$$

which by (4) equals the right-hand side of (8). If (8) is true, then by (9) again,

$$\begin{aligned}R_{2r+1}(\alpha)&=(2r+1)\alpha R_{2r}(\alpha)+(1-\alpha)\frac{d}{d\alpha}(\alpha R_{2r}(\alpha))\\&=\sum_{k=1}^{r}C_k^{2r}\{2\alpha(2k+1)+k(\alpha-1)^2\}\,\alpha^{k-1}(\alpha-1)^{2r-2k}\\&=C_1^{2r}(\alpha-1)^{2r}+C_r^{2r}2(2r+1)\alpha^r+\sum_{k=2}^{r}\{2(2k-1)\,C_{k-1}^{2r}+kC_k^{2r}\}\,\alpha^{k-1}(\alpha-1)^{2r-2k+2},\end{aligned}$$

which by (5) and (6) equals the right-hand side of (7) with r replaced by $r+1$. This completes the proof of the theorem.

Some classical formulas involving the Eulerian numbers have their counterparts in the integers C_k^n. Analogous to an identity of Worpitzky (see [3, p. 243]), we have the following theorem.

Theorem 2: For an integer $r \geq 1$:

$$x^{2r-1} = \sum_{k=1}^{r} C_k^{2r-1} \binom{x+k-1}{2k-1}. \tag{10}$$

Proof: Let Δ be the difference operator defined by $\Delta f(x) = f(x+1) - f(x)$. Following an idea of Frobenius in [5], we have by a property of S_j^n (see [3, p. 207]) and (3)

$$x^n = \sum_{j=1}^{n} j!S_j^n \binom{x}{j} = \sum_{j=1}^{n} j!S_j^n \Delta^{n-j} \binom{x}{n} = R_n(I+\Delta)\binom{x}{n}.$$

Thus, by (7),

$$x^{2r-1} = \sum_{k=1}^{r} C_k^{2r-1}(I+\Delta)^{k-1}\Delta^{2r-2k}\binom{x}{2r-1} = \sum_{k=1}^{r} C_k^{2r-1}\binom{x+k-1}{2k-1}.$$

In connection with the Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers $B_n = B_n(0)$ analogous to

$$\frac{1}{n}\{B_n(x) - B_n\} = \sum_{j=1}^{n-1} A_j^{n-1}\binom{x+j-1}{n},$$

we have the following theorem.

Theorem 3: For an integer $r \geq 1$:

$$\frac{1}{2r}\{B_{2r}(x) - B_{2r}\} = \sum_{k=1}^{r} C_k^{2r-1}\binom{x+k-1}{2k}, \tag{11}$$

$$\frac{1}{2r+1}B_{2r+1}(x) = (2x-1)\sum_{k=1}^{r}\frac{1}{2k+1}C_k^{2r}\binom{x+k-1}{2k}. \tag{12}$$

Proof: Since both sides of (11) are polynomials in x, it suffices to assume that x equals an integer $m \geq 1$. Then it follows from (10) using formulas in [3, pp. 10 and 155] that

$$\frac{1}{2r}\{B_{2r}(m) - B_{2r}\} = \sum_{j=1}^{m-1} j^{2r-1} = \sum_{k=1}^{r} C_k^{2r-1}\sum_{j=1}^{m-1}\binom{k+j-1}{2k-1} = \sum_{k=1}^{r} C_k^{2r-1}\binom{m+k-1}{2k}.$$

Similarly,

$$\begin{aligned}
\frac{1}{2r+1}B_{2r+1}(m) &= \sum_{j=1}^{m-1} j^{2r} = \sum_{k=1}^{r} C_k^{2r}\sum_{j=1}^{m-1}\frac{j}{k}\binom{k+j-1}{2k-1} \\
&= \sum_{k=1}^{r} C_k^{2r}\sum_{j=1}^{m-1}\left[\binom{k+j-1}{2k} + \binom{k+j}{2k}\right] \\
&= \sum_{k=1}^{r} C_k^{2r}\left[\binom{m+k-1}{2k+1} + \binom{m+k}{2k+1}\right] \\
&= \sum_{k=1}^{r} C_k^{2r}\frac{2m-1}{2k+1}\binom{m+k-1}{2k}.
\end{aligned}$$

As a simple and interesting consequence of Theorem 3, we derive some explicit formulas for Bernoulli numbers which may be compared with those in Theorems 5 and 6 below.

Theorem 4: For an integer $r \geq 1$:

$$B_{2r} = \sum_{k=1}^{r} (-1)^{k-1} \frac{(k-1)!k!}{(2k+1)!} C_k^{2r}, \tag{13}$$

$$B_{2r+2} = 2\sum_{k=1}^{r} (-1)^{k} \frac{k!(k+1)!}{(2k+3)!} C_k^{2r}. \tag{14}$$

Proof: We obtain (13) by differentiating both sides of (12) and then evaluating at $x=0$. Moreover, we have, by (6),

$$\begin{aligned} B_{2r+2} &= \sum_{k=1}^{r+1} (-1)^{k-1} \frac{(k-1)!k!}{(2k+1)!} k\{2(2k-1)C_{k-1}^{2r} + kC_k^{2r}\} \\ &= \sum_{k=1}^{r} (-1)^{k} \frac{k!(k+1)!}{(2k+3)!} 2\{(k+1)(2k+1) - k(2k+3)\}C_k^{2r}, \end{aligned}$$

from which (14) follows.

From the proof of Theorem 3 we have, in particular,

$$\begin{aligned} \sum_{j=1}^{m} j^{2r-1} &= \sum_{k=1}^{r} C_k^{2r-1}\binom{m+k}{2k}, \\ \sum_{j=1}^{m} j^{2r} &= (2m+1)\sum_{k=1}^{r} \frac{1}{2k+1} C_k^{2r}\binom{m+k}{2k}. \end{aligned} \tag{15}$$

We refer to [7] in which (13) and (15) have been given.

3. BERNOULLI AND EULER NUMBERS

We recall that

$$\sec t = \sum_{r=0}^{\infty} (-1)^r E_{2r} \frac{t^{2r}}{(2r)!}, \quad \tan t = \sum_{r=1}^{\infty} T_{2r-1} \frac{t^{2r-1}}{(2r-1)!},$$

where E_{2r} are known as the Euler numbers and T_{2r-1} as the tangent numbers. The Bernoulli numbers can be obtained by

$$B_{2r} = (-1)^{r-1} \frac{2r}{4^r(4^r-1)} T_{2r-1}.$$

Since

$$\sec t + \tan t = \frac{2e^{it}}{e^{2it}+1} - i\frac{e^{2it}-1}{e^{2it}+1} = 1 + (1+i)\sum_{n=1}^{\infty} H_n(i)\frac{(it)^n}{n!},$$

where $i = \sqrt{-1}$, it follows that, for $r \geq 1$,

$$E_{2r} = (1+i)H_{2r}(i), \tag{16}$$

$$T_{2r-1} = (-1)^r(1-i)H_{2r-1}(i). \tag{17}$$

Moreover, it is easy to verify that

$$T_{2r-1} = (-1)^r 2^{2r-1} H_{2r-1}(-1) = (-1)^{r-1} R_{2r-1}(-1). \tag{18}$$

See also [2, p. 257] and [3, p. 259].

Theorem 5: For an integer $r \geq 1$:

$$T_{2r-1} = \sum_{k=1}^{r} (-1)^{r-k} \frac{1}{2^{k-1}} C_k^{2r-1}, \tag{19}$$

$$T_{2r-1} = \sum_{k=1}^{r} (-1)^{r-k} 2^{2r-2k} C_k^{2r-1}; \tag{20}$$

and

$$T_{2r+1} = \sum_{k=1}^{r} (-1)^{r-k} \frac{k+1}{2^{k-1}} C_k^{2r}, \tag{21}$$

$$T_{2r+1} = \sum_{k=1}^{r} (-1)^{r-k} 2^{2r-2k+1} C_k^{2r}. \tag{22}$$

Proof: We have, by (7) and (17),

$$T_{2r-1} = (-1)^r 2 \sum_{k=1}^{r} C_k^{2r-1} \frac{i^k}{(i-1)^{2k}},$$

from which (19) follows. Moreover, we have, by (6),

$$\begin{aligned} T_{2r+1} &= \sum_{k=1}^{r+1} (-1)^{r-k+1} \frac{1}{2^{k-1}} \{2(2k-1)C_{k-1}^{2r} + kC_k^{2r}\} \\ &= \sum_{k=1}^{r} (-1)^{r-k} \frac{1}{2^k} \{2(2k+1) - 2k\} C_k^{2r}, \end{aligned}$$

from which (21) follows. We obtain (20) and (22) similarly using (18) instead.

Theorem 6: For an integer $r \geq 1$:

$$E_{2r} = \sum_{k=1}^{r} (-1)^k \frac{1}{2^{k-1}} C_k^{2r}, \tag{23}$$

$$E_{2r+2} = \sum_{k=1}^{r} (-1)^{k-1} \frac{k^2+3k+1}{2^{k-1}} C_k^{2r}. \tag{24}$$

Proof: We have, by (8) and (16),

$$E_{2r} = 2 \sum_{k=1}^{r} C_k^{2r} \frac{i^k}{(i-1)^{2k}},$$

from which (23) follows. Moreover, we have, by (6),

$$\begin{aligned} E_{2r+2} &= \sum_{k=1}^{r+1} (-1)^k \frac{k}{2^{k-1}} \{2(2k-1)C_{k-1}^{2r} + kC_k^{2r}\} \\ &= \sum_{k=1}^{r} (-1)^{k-1} \frac{1}{2^{k-1}} \{(k+1)(2k+1) - k^2\} C_k^{2r}, \end{aligned}$$

from which (24) follows.

The formulas (21) and (23) can be found in [3, p. 259] where no proofs are given. We refer to [1, pp. 479-80] for other explicit formulas for T_{2r-1} and E_{2r}.

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