ON THE NUMBER OF PERMUTATIONS WITHIN A GIVEN DISTANCE

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1. INTRODUCTION AND RESULTS

The set Π^n of all permutations of $(1, 2, \ldots, n)$, i.e., of all one-to-one mappings π from $N = \{1, 2, \ldots, n\}$ onto N, can be made to a metric space by defining

$$\|\pi = \pi'\| = \max\{|\pi(i) - \pi'(i)| : 1 \le i \le n\}.$$

This space has been studied by Lagrange [1] with emphasis on the number of points contained in a sphere with radius k around the identity, i.e.,

$$\varphi(k; n) = |\{\pi \in \Pi^n : |\pi(i) - i| \le k,\ 1 \le i \le n\}|,$$

where $|A|$ denotes the cardinality of the set A.

These numbers have been calculated in [1] for $k \in \{1, 2, 3\}$ and all $n \in \mathbb{N}$, the set of positive integers. For $k = 1$, it is fairly easy to show that $\varphi(1; n-1)$, $n \in \mathbb{N}$, $\varphi(1; 0) = 1$, is the sequence of Fibonacci numbers. For $k = 2$ and $k = 3$, the enumeration is based on quite involved recurrences. The corresponding sequences are listed in Sloane and Puffle [4] as series M1600 and M1671, respectively.

The main purpose of this note is to supplement these findings by providing a closed formula for $\varphi(k; n)$ when $k + 2 \le n \le 2k + 2$. Note that, for $n \le k + 1$, one obviously has $\varphi(k; n) = n!$; thus, the cases $n \ge 2k + 3$, $k \ge 4$, remain unresolved.

As a by-product, we obtain a formula for the permanent of specially patterned $(0, 1)$-matrices. The connection to the problem above is as follows: Let n, $k \in \mathbb{N}$, $k \le n - 1$, be fixed, and for $i \in N$, $B_i = \{j \in \mathbb{Z} : i - k \le j \le i + k\} \cap N$, where $\mathbb{Z}$ is the set of all integers.

Then $\varphi(k; n)$ is the same as the number of systems of distinct representatives for the set $\{B_1, B_2, \ldots, B_n\}$. Defining now for $i, j \in N$

$$a_{ij} = \begin{cases} 1, & j \in B_i, \\ 0, & j \notin B_i, \end{cases}$$

one has, for the permanent of the matrix $A = (a_{ij})$ (cf. Minc [2], p. 31),

$$\mathrm{Per}(A) = \varphi(k; n). \tag{1.1}$$

Remark: The recurrence formula for $\varphi(2; n)$ has also been derived by Minc using properties of permanents (see [2], p. 49, Exercise 16).

The matrix A defined in this way is symmetric and has, when $k + 2 \le n \le 2k + 2$, the block structure

$$A = \begin{pmatrix} 1_{m\times m} & 1_{m\times s} & \Delta_{m\times m} \\ 1_{s\times m} & 1_{s\times s} & 1_{s\times m} \\ \Delta^T_{m\times m} & 1_{m\times s} & 1_{m\times m} \end{pmatrix}, \tag{1.2}$$

where $m=n-1-k$, $s=2k+2-n$, $1_{a\times b}$ is the $a\times b$-matrix with all elements equal to one and $\Delta_{m\times m}$ is the $m\times m$-matrix with zeros on and above the diagonal and ones under the diagonal. For $n=2k+2$, the second row and column blocks cancel. The matrix $\Delta_{m\times m}$ has been studied by Riordan ([3], p. 211 ff.) in connection with the rook problem. Riordan proved that the numbers of ways to put r non-attacking rooks on a triangular chessboard are given by the Stirling number of the second kind. This will be crucial for the calculation of $\varphi(k;n)$ and of $\mathrm{Per}(A)$ for matrices A of a slightly more general structure than that given in (1.2). The results we will prove in Section 2 are as follows: Let S_r^n denote the Stirling numbers of the second kind, i.e., the number of ways to partition an n-set into r nonempty subsets.

Theorem 1: Let $k,n\in\mathbb{N}$, $k+2\le n\le 2k+2$, $m=n-k-1$. Then

$$\varphi(k;n)=\sum_{r=0}^{m}(-1)^{m-r}(n-2m+r)!(n-2m+r)^m S_{r+1}^{m+1}.$$

Furthermore, let the matrix A_Δ be defined as

$$A_\Delta=\begin{pmatrix} 1_{m_2\times m_1} & 1_{m_2\times m_3} & \Delta_{m_2\times m_2} \\ 1_{m_3\times m_1} & 1_{m_3\times m_3} & 1_{m_3\times m_2} \\ \Delta^T_{m_1\times m_1} & 1_{m_1\times m_3} & 1_{m_1\times m_2} \end{pmatrix}, \tag{1.3}$$

where $n\in\mathbb{N}$, $n=m_1+m_2+m_3$, $m_i\in\mathbb{N}\cup\{0\}$, $1\le i\le 3$, $\Delta_{a\times a}$ as above; for $m_i=0$, the corresponding row and column blocks cancel.

Theorem 2: Let A_Δ be defined by (1.3). Then

$$\mathrm{Per}(A_\Delta)=\sum_{r=0}^{m_1}(-1)^{m_1-r}(m_3+r)!(m_3+r)^{m_2}S_{r+1}^{m_1+1}.$$

Remarks:

(a) Since the permanent is invariant with respect to transposing a matrix and to multiplication by permutation matrices, A_Δ as given in (1.3) is only a representative of a set of matrices for which Theorem 2 holds. In particular, it follows that, for all $m_1, m_2, m_3\in\mathbb{N}\cup\{0\}$,

$$\sum_{r=0}^{m_1}(-1)^{m_1-r}(m_3+r)!(m_3+r)^{m_2}S_{r+1}^{m_1+1}=\sum_{r=0}^{m_2}(-1)^{m_2-r}(m_3+r)!(m_3+r)^{m_1}S_{r+1}^{m_2+1}.$$

Specializing further one gets, for $m_1=0$, $m_3=1$, $m_2+1=m$, the well-known relation

$$1=\sum_{r=1}^{m}(-1)^{m-r}r!\,S_r^m.$$

(b) Since the matrix A given in (1.2) is a special case of the matrix A_Δ, in view of (1.1), Theorem 1 is a special case of Theorem 2. Therefore, we have to prove only Theorem 2.

2. PROOFS

By a suitable identification of the rook problem discussed in Riordan [3], chapters 7 and 8, with the problem considered here, part of the proof of Theorem 2 could be derived from results in

[3]. In view of a certain consistence of the complete proof, we prefer however to develop the necessary details from the beginning.

The problem of determining $\varphi(k;n)$ can be seen as a problem of finding the cardinality of an intersection of unions of sets. We will do this by applying the principle of inclusion and exclusion to its complement. Therefore, the sets $\Pi_{ij}^n = \{\pi \in \Pi^n : \pi(i) = j\}$, $i, j \in N = \{1, 2, \dots, n\}$ are relevant. Let $\mathcal{P}_k(J)$ for $J \subset \mathbb{N}$ denote the set of all $I \subset J$ with $|I| = k$ and $\mathbb{N}_{\neq}^k$ the set of all k-tuples in $\mathbb{N}^k$ with pairwise different components. For $k, n \in \mathbb{N}$, $k \le n$, $(i_1, i_2, \dots, i_k) \in \mathbb{N}_{\neq}^k \cap N^k$, and $j_v \in N$, $1 \le v \le k$, one obviously has

$$\left|\bigcap_{v=1}^{k} \Pi_{i_v j_v}^n\right| = \begin{cases}(n-k)! & \text{if } (j_1, j_2, \dots, j_k) \in \mathbb{N}_{\neq}^k,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, one gets from the principle of inclusion and exclusion that, for $k, n \in \mathbb{N}$, $k \le n$, $J \subset N$ with $|J| = k$, and $B_i \subset N$, $i \in J$,

$$\left|\bigcup_{i \in J} \bigcup_{j \in B_i} \Pi_{ij}^n\right| = \sum_{r=1}^{k} (-1)^{r-1} (n-r)! \sum_{I \in \mathcal{P}_r(J)} \left|\{(j_1, \dots, j_r) \in \mathbb{N}_{\neq}^r : j_i \in B_i \ \forall i \in I\}\right|. \tag{2.1}$$

For the sets on the right-hand side of (2.1), it holds that

$$\left|\{(j_1, \dots, j_r) \in \mathbb{N}_{\neq}^r : j_i \in B_i \ \forall_i \in I\}\right| = \frac{1}{(n-k)!}\left|\bigcap_{i \in J}\{\pi \in \Pi^n : \pi(i) \in B_i\}\right|. \tag{2.2}$$

For $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, $k \le n$, $B_1, B_2, \dots, B_n \subset N$, let

$$R_k^n(B_1, \dots, B_n) = \begin{cases}\displaystyle\sum_{j \in \mathcal{P}_k(N)} \left|\{(j_1, \dots, j_k) \in \mathbb{N}_{\neq}^k : j_i \in B_i \ \forall i \in J\}\right|, & \text{for } k \ge 1,\\ 1, & \text{for } k = 0.\end{cases} \tag{2.3}$$

[If one considers a chessboard on which pieces may be placed only on positions (i, j) for which $j \in B_i$, then $R_k^n(B_1, \dots, B_n)$ is the number of ways of putting k non-attacking rooks on this board.]

Lemma 1: Let $k, n \in \mathbb{N}$, $j \le n$, $B_i \subset N$ for $i \in N$. Then it holds that

$$\sum_{J \in \mathcal{P}_k(N)} \left|\bigcup_{i \in J} \bigcup_{j \in B_i} \Pi_{ij}^n\right| = \sum_{r=1}^{k} (-1)^{r-1} (n-r)! \binom{n-r}{k-r} R_r^n(B_1, \dots, B_n).$$

Proof: With the help of (2.1), one gets

$$\begin{aligned}
&\sum_{J \in \mathcal{P}_k(N)} \left|\bigcup_{i \in J} \bigcup_{j \in B_i} \Pi_{ij}^n\right| \\
&= \sum_{r=1}^{k} (-1)^{r-1} (n-r)! \sum_{J \in \mathcal{P}_k(N)} \sum_{I \in \mathcal{P}_r(J)} \left|\{(j_1, \dots, j_r) \in \mathbb{N}_{\neq}^r : j_i \in B_i \ \forall i \in I\}\right| \\
&= \sum_{r=1}^{k} (-1)^{r-1} (n-r)! \sum_{I \in \mathcal{P}_r(N)} \left|\{(j_1, \dots, j_r) \in \mathbb{N}_{\neq}^r : j_i \in B_i \ \forall i \in I\}\right| \left|\{J \in \mathcal{P}_k(N) : I \subset J\}\right| \\
&= \sum_{r=1}^{k} (-1)^{r-1} (n-r)! \binom{n-r}{k-r} R_r^n(B_1, \dots, B_n). \ \square
\end{aligned}$$

In the next lemma it is shown how the numbers $R_k^n(B_1, \dots, B_n)$ are related to $R_k^n(B_1^c, \dots, B_n^c)$, where B_i^c denotes the complement of B_i w.r.t. N. (In terms of the rook problem, one thus considers the complement of the chessboard.) The lemma is equivalent to Theorem 2 in Riordan ([3], p. 180).

Lemma 2: Let $k, n \in \mathbb{N}$, $k \le n$, $B_i \subset N$, $i \in N$. Then it holds that

$$R_k^n(B_1, \dots, B_n) = \sum_{r=0}^{k} (-1)^r (k-r)! \binom{n-r}{n-k} \binom{n-r}{k-r} R_r^n(B_1^c, \dots, B_n^c).$$

Proof: By (2.2) and (2.3), one has

$$\begin{aligned}(n-k)! R_k^n(B_1, \dots, B_n) &= \sum_{J \in \mathcal{P}_k(N)} \left| \bigcap_{i \in J} \{\pi \in \Pi^n : \pi(i) \in B_i\} \right| \\ &= \sum_{J \in \mathcal{P}_k(N)} \left(n! - \left| \bigcup_{i \in J} \bigcup_{j \in B_i^c} \Pi_{ij}^n \right| \right).\end{aligned}$$

The assertion then follows with the help of Lemma 1. □

Lemma 2 will become useful for calculating $\mathrm{Per}(A_\Delta)$ in the following manner: Let $A_\Delta = (a_{ij})$ and put $B_i = \{j \in N : a_{ij} = 1\}$. Since by (2.2) and (2.3)

$$\mathrm{Per}(A_\Delta) = \sum_{\pi \in \Pi^n} \prod_{i=1}^{n} a_{i,\pi(i)} = \left| \left\{ \pi \in \Pi^n : \prod_{i=1}^{n} a_{i,\pi(i)} = 1 \right\} \right| = R_n^n(B_1, \dots, B_n),$$

one obtains from Lemma 2 that

$$\mathrm{Per}(A_\Delta) = \sum_{r=0}^{n} (-1)^r (n-r)! R_r^n(B_1^c, \dots, B_n^c). \tag{2.4}$$

The matrix corresponding to $B_1^c, \dots, B_n^c$ is $\overline{A}_\Delta = 1_{n \times n} - A_\Delta$, which is easier to handle because it has mainly blocks of zero-matrices. A further simplification is obtained by considering instead of $\overline{A}_\Delta$ the matrix

$$\hat{A}_\Delta = \begin{pmatrix} \hat{\Delta}_{m_1 \times m_1} & 0_{m_1 \times m_2} & 0_{m_1 \times m_3} \\ 0_{m_2 \times m_1} & \hat{\Delta}_{m_2 \times m_2} & 0_{m_2 \times m_3} \\ 0_{m_3 \times m_1} & 0_{m_3 \times m_2} & 0_{m_3 \times m_3} \end{pmatrix}, \tag{2.5}$$

where $\hat{\Delta}_{a \times a} = 1_{a \times a} - \Delta_{a,a}^T \cdot \hat{A}_\Delta$ is obtained from $\overline{A}_\Delta$ by suitable permutations of rows and columns. By Remark (a) one has $\mathrm{Per}(\overline{A}_\Delta) = \mathrm{Per}(\hat{A}_\Delta)$.

Now we turn to the special structure related to the matrices of the form $\hat{\Delta}_{m \times m}$, that is, we consider $B_i = \{1, 2, \dots, i\}$, $i \in N_m = \{1, 2, \dots, m\}$. One can easily show by induction on k that

$$\left| \{(j_1, \dots, j_k) \in \mathbb{N}_{\neq}^k : j_\nu \in D_\nu, 1 \le \nu \le k\} \right| = \prod_{\nu=1}^{k} (|D_\nu| + 1 - \nu)$$

if $k, m \in \mathbb{N}$, $k \le m$, and $D_1, \dots, D_k \subset N_m$ such that $D_\nu \subset D_{\nu+1}$, $1 \le \nu < k$, so that

$$R_k^m(B_1, \dots, B_m) = \sum_{1 \le i_1 < \cdots < i_k \le m} \prod_{\nu=1}^{k} (i_\nu + 1 - \nu). \tag{2.6}$$

We denote the right-hand side of (2.6) by α_k^m, $1 \le k \le m$, $\alpha_0^m = 1$, $\alpha_k^m = 0$, for $k < 0$ or $k > m$.

Lemma 3: For α_k^m defined as above, it holds that

(a) $\alpha_k^m = \alpha_k^{m-1} + (m+1-k)\alpha_{k-1}^{m-1}$ for all $k \in \mathbb{Z}$, $m \in \mathbb{N}$, $m \ge 2$.

(b) $\alpha_k^m = S_{m+1-k}^{m+1}$ for all $m \in \mathbb{N}$, $k \in \mathbb{N}_\cup\{0\}$, $k \le m$.

Proof: Part (a) follows immediately from the definition of α_k^m. Assertion (b) obviously holds true for $m = 1$. Since the Stirling numbers of the second kind satisfy the recursion $S_k^m = S_{k-1}^{m-1} + kS_k^{m-1}$, the assertion is a consequence of (a). □

It now follows from Lemma 3 and (2.6) that, for $B_i = \{1, 2, \dots, i\}$, $1 \le i \le m$,

$$R_k^m(B_1, \dots, B_m) = \begin{cases} S_{m+1-k}^{m+1}, & \text{for all } m \in \mathbb{N}, k \in \mathbb{N}_\cup\{0\}, k \le m, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

To deal with the two Δ-blocks of the matrix $\hat{A}_\Delta$, the following lemma is helpful.

Lemma 4: Let $m_1, m_2, n \in \mathbb{N}$, $n \ge m_1 + m_2$, and $C_1, C_2, \dots, C_n \subset N$ such that:

(a) $C_i \subset \{1, 2, \dots, m_1\}$, $1 \le i \le m_1$;

(b) $C_i \subset \{m_1 + 1, \dots, m_1 + m_2\}$, $m_1 + 1 \le i \le m_1 + m_2$;

(c) $C_i = \emptyset$, $m_1 + m_2 + 1 \le i \le n$.

Furthermore, let $D_i = \{j \in \{1, \dots, m_2\} : j + m_1 \in C_{i+m_1}\}$, $1 \le i \le m_2$. Then it holds that

$$R_k^n(C_1, \dots, C_n) = \begin{cases} \sum_{\nu=0}^{k} R_\nu^{m_1}(C_1, \dots, C_{m_1}) R_{k-\nu}^{m_2}(D_1, \dots, D_{m_2}), & 0 \le k \le m_1 + m_2, \\ 0, & m_1 + m_2 + 1 \le k \le n. \end{cases}$$

Proof: Let $N_1 = \{1, \dots, m_1\}$, $N_2 = \{m_1 + 1, \dots, m_1 + m_2\}$, $N_3 = \{m_1 + m_2 + 1, \dots, n\}$ and, for $J \in \mathcal{P}_k(N)$, $f_k(J) = \left|\{(j_1, \dots, j_k) \in \mathbb{N}_{\neq}^k : j_i \in C_i \ \forall i \in J\}\right|$. Since $C_i = \emptyset$ for $i \in N_3$, one has $f_k(J) = 0$ if $J \in \mathcal{P}_k(N)$ and $J \cap N_3 \ne \emptyset$. This implies

$$R_k^n(C_1, \dots, C_n) = \sum_{r=0}^{k} \sum_{J_1 \in \mathcal{P}_r(N_1)} \sum_{J_2 \in \mathcal{P}_{k-r}(N_2)} f_k(J_1 \cup J_2).$$

Since $\left(\bigcup_{i \in N_1} C_i\right) \cap \left(\bigcup_{i \in N_2} C_i\right) = \emptyset$ one has, for $J_1 \in \mathcal{P}_r(N_1)$, $J_2 \in \mathcal{P}_{k-r}(N_2)$, that

$$f_k(J_1 \cup J_2) = \left|\{(j_1, \dots, j_r) \in \mathbb{N}_{\neq}^r,\ j_i \in C_i \ \forall i \in J_1\}\right| \left|\{(j_{r+1}, \dots, j_k) \in \mathbb{N}_{\neq}^{k-r} : j_i \in C_i \ \forall i \in J_2\}\right|$$

The assertion then follows from

$$R_r^{m_1}(C_1, \dots, C_{m_1}) = \sum_{J_1 \in \mathcal{P}_r(N_1)} \left|\{(j_1, \dots, j_r) \in \mathbb{N}_{\neq}^r : j_i \in C_i \ \forall i \in J_1\}\right|$$

and

$$R_{k-r}^{m_2}(D_1, \dots, D_{m_2}) = \sum_{J_2 \in \mathcal{P}_{k-r}(N_2)} \left|\{(j_{r+1}, \dots, j_k) \in \mathbb{N}_{\neq}^{k-r} : j_i \in C_i \ \forall i \in J_2\}\right|. \ \square$$

Finally, the following identity will become useful:

$$\sum_{r=0}^{m}(-1)^r(n-r)!\,S_{m+1-r}^{m+1}=(n-m)!(n-m)^m \text{ for } m,n\in\mathbb{N}\cup\{0\},\ n\geq m. \tag{2.8}$$

Identity (2.8) can easily be proved by induction on m using the recurrence formula for the Stirling numbers. Now we are ready to prove Theorem 2. Consider the matrix $\hat{A}_\Delta=(\hat{a}_{ij})$ defined in (2.5). Putting

$$C_i=\begin{cases}\{1,\dots,i\}, & 1\leq i\leq m_1,\\ \{m_1+1,\dots,m_1+i-m_1\}, & m_1+1\leq i\leq m_1+m_2,\\ \emptyset, & m_1+m_2+1\leq i\leq n,\end{cases}$$

one has $\hat{a}_{ij}=1$ if and only if $j\in C_i$. Note that for $C_1,\dots,C_n$ the assumptions of Lemma 4 are satisfied and that $D_i=\{1,\dots,i\}$ for $1\leq i\leq m_2$. Put $n-m_1-m_2=m_3$. Then, from (2.4), Lemma 4, (2.7), and (2.8), one gets that

$$\begin{aligned}
\mathrm{Per}(A_\Delta)&=\sum_{r=0}^{n}(-1)^r(n-r)!\,R_r^n(C_1,\dots,C_n)\\
&=\sum_{r=0}^{m_1+m_2}(-1)^r(n-r)!\sum_{\nu=0}^{r}R_\nu^{m_1}(C_1,\dots,C_{m_1})\,R_{r-\nu}^{m_2}(D_1,\dots,D_{m_2})\\
&=\sum_{r=0}^{m_1+m_2}(-1)^r(n-r)!\sum_{\nu=0}^{r}S_{m_1+1-\nu}^{m_1+1}S_{m_2+1-r+\nu}^{m_2+1}=\sum_{\nu=0}^{m_1+m_2}S_{m_1+1-\nu}^{m_1+1}\sum_{r=\nu}^{m_1+m_2}(-1)^r(n-r)!\,S_{m_2+1-r+\nu}^{m_2+1}\\
&=\sum_{\nu=0}^{m_1}S_{m_1+1-\nu}^{m_1+1}\sum_{r=0}^{m_2}(-1)^{r+\nu}(n-r-\nu)!\,S_{m_2+1-r}^{m_2+1}=\sum_{\nu=0}^{m_1}(-1)^\nu S_{m_1+1-\nu}^{m_1+1}(n-\nu-m_2)!(n-\nu-m_2)^{m_2}\\
&=\sum_{\nu=0}^{m_1}(-1)^{m_1-\nu}(n-m_1+\nu-m_2)!(n-m_1+\nu-m_2)^{m_2}S_{\nu+1}^{m_1+1}\\
&=\sum_{\nu=0}^{m_1}(-1)^{m_1-\nu}(m_3+\nu)!(m_3+\nu)^{m_2}S_{\nu+1}^{m_1+1}.
\end{aligned}$$

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