GENERALIZED FIBONACCI SEQUENCES AND LINEAR CONGRUENCES

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(*Submitted September 2000-Final Revision July 2001*)

1. INTRODUCTION

There exists a very wide literature about the generalized Fibonacci sequences (see, e.g., [3], where interesting applications to number theory are also shown, and [2], where such sequences are treated as a particular case of a more general class of sequences of numbers). In this paper we start by defining some particular generalized Fibonacci sequences (denoted by $\{U_n(c-1,-c)\}_{n\in\mathbb{N}}$, $c\in\mathbb{N}$) and by studying their properties. In particular, we find interesting relations between a generic term $U_n(c-1,-c)$, $n\in\mathbb{N}$, and $U_{n+1}(c-1,-c)$ and show a nice connection between the numbers $U_n(c-1,-c)$ and their expression in the c-ary enumeration system. After this, we give an estimate of the value of the logarithm of $U_n(c-1,-c)$ on the basis c.

Successively, we apply the properties of the sequences $\{U_n(c-1,-c)\}_{n\in\mathbb{N}}$ to the study of the number of solutions of linear equations in $\mathbb{Z}_r$, $r\in\mathbb{N}$.

Finally, we briefly show the principal characteristics of another class of generalized Fibonacci sequences, $\{U_n(c+1,c)\}_{n\in\mathbb{N}}$, $c\in\mathbb{N}\setminus\{1\}$.

2. GENERALIZED FIBONACCI SEQUENCES: THE SEQUENCES $\{U_n(c-1,-c)\}_{n\in\mathbb{N}}$

For each pair (h,k), $h,k\in\mathbb{C}$ of complex numbers such that $k(h^2-4k)\neq 0$, we denote by $\{U_n(h,k)\}_{n\in\mathbb{N}}$ the generalized Fibonacci sequence defined as follows:

$$\forall n\in\mathbb{N},\ n\geq 2,\ U_n(h,k)=hU_{n-1}(h,k)-kU_{n-2}(h,k),\ U_0(h,k)=0,\ U_1(h,k)=1.$$

An explicit expression of the n^{th} term of $\{U_n(h,k)\}_{n\in\mathbb{N}}$ for generic $n\in\mathbb{N}\cup\{0\}$ is given by the Binet formula

$$U_n(h,k)=\frac{\alpha^n-\beta^n}{\alpha-\beta},$$

where

$$\alpha=\frac{h+\sqrt{h^2-4k}}{2}\quad\text{and}\quad\beta=\frac{h-\sqrt{h^2-4k}}{2}$$

are the distinct roots of the polynomial $x^2-hx+k\in\mathbb{C}[x]$, called the characteristic polynomial of the sequence. Moreover, for every integer $n\in\mathbb{N}\cup\{0\}$, we have

$$\alpha\cdot\frac{\alpha^n-\beta^n}{\alpha-\beta}+\beta^n=\frac{\alpha^{n+1}-\alpha\beta^n+\alpha\beta^n-\beta^{n+1}}{\alpha-\beta}=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}.$$

We then obtain

$$\forall n\in\mathbb{N}\cup\{0\},\quad \alpha\cdot U_n(h,k)+\beta^n=U_{n+1}(h,k). \tag{1}$$

As the role played by α and β in the Binet formulas is symmetric, the following equalities are also true:

$$\forall n \in \mathbb{N} \cup \{0\}, \quad \beta \cdot U_n(h, k) + \alpha^n = U_{n+1}(h, k). \tag{2}$$

As a particular case, let us consider now the generalized Fibonacci sequences of the form $\{U_n(c-1, -c)\}_{n \in \mathbb{N}}$, c being a positive integer; from the equalities $h = c-1$ and $k = -c$, we easily obtain $\alpha = c$ and $\beta = -1$. Then, for all $n \in \mathbb{N} \cup \{0\}$, from the Binet formula we have

$$U_n(c-1, -c) = \frac{c^n - (-1)^n}{c+1},$$

while equalities (1) and (2) show, respectively, that

$$\forall n \in \mathbb{N} \cup \{0\}, \; U_{n+1}(c-1, -c) = cU_n(c-1, -c) + (-1)^n, \tag{3}$$

and

$$\forall n \in \mathbb{N} \cup \{0\}, \; U_n(c-1, -c) + U_{n+1}(c-1, -c) = c^n. \tag{4}$$

The first terms of some of such generalized Fibonacci sequences, corresponding to fixed values of c, are:

$$\{U_n(0, -1)\}_{n \in \mathbb{N}}: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots;$$
$$\{U_n(1, -2)\}_{n \in \mathbb{N}}: 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, \ldots;$$
$$\{U_n(2, -3)\}_{n \in \mathbb{N}}: 0, 1, 2, 7, 20, 61, 182, 547, 1640, 4921, \ldots;$$
$$\{U_n(3, -4)\}_{n \in \mathbb{N}}: 0, 1, 3, 13, 51, 205, 819, 3277, 13107, 52429, \ldots;$$
$$\{U_n(5, -6)\}_{n \in \mathbb{N}}: 0, 1, 5, 31, 185, 1111, 6665, 39991, 239945, \ldots.$$

3. $\{U_n(c-1, -c)\}_{n \in \mathbb{N}}$ $(c \geq 2)$ IN THE c-ARY ENUMERATION SYSTEM

Theorem: Let $c \geq 2$ be a fixed integer; then, for each fixed integer $m \geq 2$, the two following assertions are equivalent:

(a) $\exists n \in \mathbb{N}: m = U_n(c-1, -c)$;

(b) in the c-ary enumeration system, the expression of m is either of the form $(c-1)0(c-1)\ldots0(c-1)$ or of the form $(c-1)0(c-1)\ldots0(c-1)1$.

Moreover, when for a given m the two assertions are satisfied, we have $m = U_{t+1}(c-1, -c)$, where t denotes the number of digits of m which appear when it is written in the c-ary enumeration system.

The theorem can be proven by noticing that, for every $n \in \mathbb{N} \cup \{0\}$, we have the recursion $U_{n+1}(c-1, -c) = cU_n(c-1, -c) + (-1)^n$. Hence, if (a) is satisfied, assertion (b) straightforwardly follows by induction from the first few terms:

$$U_2(c-1, -c) = c \cdot 1 - 1 = \mathbf{c-1};$$
$$U_3(c-1, -c) = c \cdot (c-1) + 1 = \mathbf{10} \cdot (\mathbf{c-1}) + \mathbf{1} = (\mathbf{c-1})\mathbf{0} + \mathbf{1} = (\mathbf{c-1})\mathbf{1};$$
$$U_4(c-1, -c) = c \cdot U_3(c-1, -c) - 1 = \mathbf{10} \cdot [(\mathbf{c-1})\mathbf{1}] - \mathbf{1} = (\mathbf{c-1})\mathbf{10} - \mathbf{1} = (\mathbf{c-1})\mathbf{0}(\mathbf{c-1});$$
$$U_5(c-1, -c) = c \cdot U_4(c-1, -c) + 1 = \mathbf{10} \cdot [(\mathbf{c-1})\mathbf{0}(\mathbf{c-1})] + \mathbf{1} = (\mathbf{c-1})\mathbf{0}(\mathbf{c-1})\mathbf{0} + \mathbf{1} = (\mathbf{c-1})\mathbf{0}(\mathbf{c-1})\mathbf{1};$$

$$\begin{aligned}U_6(c-1,-c) &= c\cdot U_5(c-1,-c)-1 = \mathbf{10}\cdot[(\mathbf{c}-\mathbf{1})\mathbf{0}(\mathbf{c}-\mathbf{1})\mathbf{1}]-\mathbf{1}\\ &= (\mathbf{c}-\mathbf{1})\mathbf{0}(\mathbf{c}-\mathbf{1})\mathbf{10}-\mathbf{1} = (\mathbf{c}-\mathbf{1})\mathbf{0}(\mathbf{c}-\mathbf{1})\mathbf{0}(\mathbf{c}-\mathbf{1}).\end{aligned}$$

(For the sake of clarity, the convention was adopted of writing the c-ary expressions in boldface characters; the dot denotes multiplication.) Conversely, if (b) is satisfied, m is clearly seen to be a term of the sequence $\{U_n(c-1,-c)\}_{n\in\mathbb{N}}$ by applying a finite number of times the recursion $U_{n+1}(c-1,-c)=cU_n(c-1,-c)+(-1)^n$, and assertion (a) follows.

Moreover, it is clear that, for every $n\geq 2$, the number of digits of $U_{n+1}(c-1,-c)$ when it is written in the c-ary system is one unit larger than the number of digits of $U_n(c-1,-c)$ when it is expressed in the same system. Since in the c-ary system the number $U_2(c-1,-c)$ is expressed by the only digit $c-1$, the second part of the theorem follows by induction.

4. AN ESTIMATE OF $\log_c(U_n(c-1,-c))$ $(c\geq 2, n\geq 1)$

For any $c\geq 2$ and $n\geq 1$, we know that

$$U_n(c-1,-c)=\frac{c^n-(-1)^n}{c+1};$$

hence, we have $\log_c(U_n(c-1,-c))=\log_c(c^n-(-1)^n)-\log_c(c+1)$, which is equal to

$$\log_c\left[c^n\left(1-\frac{(-1)^n}{c^n}\right)\right]-\log_c\left[c\left(1+\frac{1}{c}\right)\right]=n-1+\log_c\left(1-\frac{(-1)^n}{c^n}\right)-\log_c\left(1+\frac{1}{c}\right).$$

Now we suppose c fixed and consider $\log_c(U_n(c-1,-c))$ as a function of n. Since

$$\frac{\ln(1+y)}{y}=1+o(1) \text{ as } y\to 0,$$

we have $\ln(1+y)=y+o(y)$ $(y\to 0)$; $\log_c(1+y)=\frac{y}{\ln c}+o(y)$ $(y\to 0)$. Then, for $n\to+\infty$, we can write

$$\log_c\left(1-\frac{(-1)^n}{c^n}\right)=\frac{(-1)^{n-1}}{c^n\ln c}+o\left(\frac{1}{c^n}\right) \quad (n\to+\infty).$$

On the other hand, for every positive real number x, the following inequalities hold: $0<\ln(1+x)<x$; hence, we have $0<\log_c(1+x)<\frac{x}{\ln c}$. Taking $x=\frac{1}{c}$, we obtain

$$0<\log_c\left(1+\frac{1}{c}\right)<\frac{1}{c\ln c}.$$

Then, from the above equalities we have, when setting $\gamma(c)=\log_c(1+\frac{1}{c})$, the approximation of $\log_c(U_n(c-1,-c))$ holding for n large,

$$\begin{aligned}\log_c(U_n(c-1,-c)) &= n-1+\log_c\left(1-\frac{(-1)^n}{c^n}\right)-\log_c\left(1+\frac{1}{c}\right)\\ &= n-1-\gamma(c)+\frac{(-1)^{n-1}}{c^n\ln c}+o\left(\frac{1}{c^n}\right) \quad (n\to+\infty),\end{aligned}$$

where $0<\gamma(c)<\frac{1}{c\ln c}$.

5. LINEAR EQUATIONS IN $\mathbb{Z}_r$ AND THEIR RELATION WITH THE SEQUENCES $\{U_n(c-1,-c)\}_{n\in\mathbb{N}}$

We consider the problem of finding the elements $(x_1; x_2; \ldots; x_k) \in (\mathbb{Z}_r)^k$ which satisfy the congruence equation

$$\sum_{j=1}^{k} x_j \equiv a \pmod{r}, \tag{5}$$

and the constraining equalities

$$\gcd(x_j, r) = d_j;\ j = 1, 2, \ldots, k, \tag{6}$$

where r and k are fixed positive integers, r is odd, $a \in \mathbb{Z}_r$, and $d_1, d_2, \ldots, d_k$ are k divisors (not necessarily distinct) of r. Let us pose, for each prime divisor p of r, $b_p = \#(\{j, 1 \le j \le k : p \nmid d_j\})$, and let us assume that, for each p, $b_p \ge 2$.

Starting from formulas which give the total number N_a of solutions of the above problem (see [1], eq. (3.37), and [4], ex. 3.8, p. 138), replacing in such formulas Ramanujan sums by their expressions as given by Hölder's equalities, i.e.,

$$\forall m, n \in \mathbb{N}, c(m; n) = \sum_{\substack{j=1 \\ \gcd(j,n)=1}}^{n} (e^{2\pi i/n})^{jm} = \frac{\varphi(n)}{\varphi(n/\gcd(n,m))} \cdot \mu(n/\gcd(n,m)),$$

φ and μ being, respectively, Euler's and Möbius' functions (see [5]), and then using basic properties of φ and μ and applying (in reverse order) the distributive property of the product with respect to the sum, gives rise to the following equality:

$$N_a = \frac{\varphi(r/d_1)\varphi(r/d_2)\ldots\varphi(r/d_k)}{r} \cdot P_a, \tag{7}$$

where

$$P_a = \prod_{p|r,\, p\nmid a}\left[1 - \frac{(-1)^{b_p}}{(p-1)^{b_p}}\right] \cdot \prod_{p|r,\, p|a}\left[1 - \frac{(-1)^{b_p-1}}{(p-1)^{b_p-1}}\right]. \tag{8}$$

The latter formula can be found in [5] for the special case $d_1 = d_2 = \cdots = d_k = 1$ only. Compare equalities (7) and (8) also with [6].

Now we want to rewrite equality (8) in terms of the generalized Fibonacci sequences that we treated in the previous sections. First, we observe that, for each prime divisor p of r, by applying the Binet formula to the terms of $\{U_n(c-1,-c)\}_{n\in\mathbb{N}}$ in the case in which $c = p-1$, we have, for each nonnegative integer n,

$$U_n(p-2, 1-p) = \frac{(p-1)^n - (-1)^n}{p},$$

i.e., $pU_n(p-2, 1-p) = (p-1)^n - (-1)^n$. Hence, from (8), we obtain

$$\begin{aligned} P_a &= \prod_{p|r,\, p\nmid a}\left[\frac{(p-1)^{b_p} - (-1)^{b_p}}{(p-1)^{b_p}}\right] \cdot \prod_{p|r,\, p|a}\left[\frac{(p-1)^{b_p-1} - (-1)^{b_p-1}}{(p-1)^{b_p-1}}\right] \\ &= \prod_{p|r,\, p\nmid a}\left[\frac{p \cdot U_{b_p}(p-2, 1-p)}{(p-1)^{b_p}}\right] \cdot \prod_{p|r,\, p|a}\left[\frac{p \cdot U_{b_p-1}(p-2, 1-p)}{(p-1)^{b_p-1}}\right] \end{aligned}$$

$$=\prod_{p\mid r}\left[\frac{p}{(p-1)^{b_p-1}}\right]\cdot\prod_{p\mid r,\, p\nmid a}\left[\frac{U_{b_p}(p-2,1-p)}{p-1}\right]\cdot\prod_{p\mid r,\, p\mid a}U_{b_p-1}(p-2,1-p). \tag{9}$$

Now let us fix a prime divisor q of r and let u be a residue class in $\mathbb{Z}_r$ such that $q\nmid u$. We want to calculate the ratio of P_{qu} to P_u. From expression (9) of P_a for generic a, comparing the case in which $a=qu$ with the case in which $a=u$, we immediately obtain

$$\frac{P_{qu}}{P_u}=\frac{U_{b_q-1}(q-2,1-q)}{U_{b_q}(q-2,1-q)/(q-1)}=\frac{(q-1)U_{b_q-1}(q-2,1-q)}{U_{b_q}(q-2,1-q)}. \tag{10}$$

Moreover, from (3), taking $c=q-1$ and $n=b_q-1$, we obtain

$$U_{b_q}(q-2,1-q)=(q-1)U_{b_q-1}(q-2,1-q)+(-1)^{b_q-1},$$

i.e., $(q-1)U_{b_q-1}(q-2,1-q)=U_{b_q}(q-2,1-q)+(-1)^{b_q}$, and hence

$$\frac{P_{qu}}{P_u}=\frac{U_{b_q}(q-2,1-q)+(-1)^{b_q}}{U_{b_q}(q-2,1-q)}=1+\frac{(-1)^{b_q}}{U_{b_q}(q-2,1-q)}. \tag{11}$$

Equations (11) show that the ratio P_{qu}/P_u depends on q, but is independent of u. They also show that, when b_q is even, then $P_{qu}>P_u$, while when b_q is odd, then $P_{qu}<P_u$. This means that a sum having an even number of addenda which are not multiples of q tends to favor as possible results the multiples of q, while a sum having an odd number of addenda which are not multiples of q tends to favor the numbers which are not multiples of q. Moreover, since r is odd (which implies $q\geq 3$) and for $c\geq 2$ the integer $U_n(c-1,-c)$ tends to infinity as $n\to+\infty$, equations (11) show that the greater b_q, the nearer one to another are the values of P_{qu} and P_u. This means that if in a sum there are many addenda which are not multiples of q, then the sum tends to favor significantly neither the multiples of q nor the integers which are not multiples of q. More generally, in view of (7) and (8), the distribution in $\mathbb{Z}_r$ of the values of the expression $\sum_{j=1}^k x_j$ as $x_1, x_2, \ldots, x_k$ vary in $\mathbb{Z}_r^*$, tends to be a uniform distribution as k tends to infinity (because P_a tends to 1 and N_a becomes independent of a).

Furthermore, if $q^2\nmid r$, then for each residue class a in $\mathbb{Z}_r$ which is a multiple of q, there exist exactly $q-1$ classes u in $\mathbb{Z}_r$ not multiples of q such that $a\equiv qu \pmod r$. In this case, from equations (10), dividing P_{qu}/P_u by $q-1$, we obtain the number

$$\frac{U_{b_q-1}(q-2,1-q)}{U_{b_q}(q-2,1-q)}, \tag{12}$$

which, being independent of a, can be considered as the ratio of the number of the strings $(x_1; x_2; \ldots; x_k)$ such that $q\mid\sum_{j=1}^k x_j$ to the number of the strings $(x_1; x_2; \ldots; x_k)$ such that $q\nmid\sum_{j=1}^k x_j$.

We now give an example of what was discussed in this section. Let the following problem be assigned:

$$\sum_{j=1}^{7}x_j\equiv a \pmod 3,\quad \gcd(x_j,3)=1 \text{ for } j=1,2,\ldots,7.$$

We want to calculate the ratio N_0/N_1.

By taking $q=3$ and $u=1$, we have $b_q=7$ and then, by (11), we can write

$$\frac{N_0}{N_1}=\frac{N_3}{N_1}=\frac{P_3}{P_1}=1+\frac{(-1)^7}{U_7(1,-2)}=1-\frac{1}{43}=\frac{42}{43}.$$

To obtain the ratio of the number of strings $(x_1; x_2; \ldots; x_7) \in (\mathbb{Z}_3^*)^7$ such that $3 \mid \sum_{j=1}^{7} x_j$ to the number of strings $(x_1; x_2; \ldots; x_7) \in (\mathbb{Z}_3^*)^7$ such that $3 \nmid \sum_{j=1}^{7} x_j$, we use expression (12) and find that this ratio is equal to $\frac{U_6(1,-2)}{U_7(1,-2)}$, i.e., to $\frac{21}{43}$.

6. THE SEQUENCES $\{U_n(c+1, c)\}_{n\in\mathbb{N}}$

Another interesting class of generalized Fibonacci sequences is the set $\{U_n(c+1,c)\}_{n\in\mathbb{N}}$, i.e., of the sequences whose characteristic polynomial has c and 1 as roots, c being a positive integer not equal to 1.

For all $n \in \mathbb{N} \cup \{0\}$, we have the Binet formulas

$$U_n(c+1,c)=\frac{c^n-1}{c-1}; \text{ then } \forall n \in \mathbb{N},\ U_n(c+1,c)=c^{n-1}+c^{n-2}+\cdots+c+1.$$

Some examples of such sequences are:

$$\{U_n(3,2)\}_{n\in\mathbb{N}}: 0, 1, 3, 7, 15, 31, 63, 127, \ldots;$$
$$\{U_n(4,3)\}_{n\in\mathbb{N}}: 0, 1, 4, 13, 40, 121, 364, 1093, \ldots;$$
$$\{U_n(5,4)\}_{n\in\mathbb{N}}: 0, 1, 5, 21, 85, 341, 1365, 5461, \ldots;$$
$$\{U_n(6,5)\}_{n\in\mathbb{N}}: 0, 1, 6, 31, 156, 781, 3906, 19531, \ldots.$$

From equalities (1) and (2) we have, respectively,

$$\forall n \in \mathbb{N} \cup \{0\},\ U_{n+1}(c+1,c)=cU_n(c+1,c)+1$$

and

$$\forall n \in \mathbb{N} \cup \{0\},\ U_{n+1}(c+1,c)=U_n(c+1,c)+c^n.$$

For a fixed c, it is clear that the terms of $\{U_n(c+1,c)\}_{n\in\mathbb{N}}$, if we exclude the first term 0, are exactly the integers which in the c-ary system are written in the form $11\ldots1$. Moreover, for each $n \in \mathbb{N}$, the number of digits "1" that appear in the expression of $U_n(c+1,c)$ in the c-ary system is n.

For any $c \geq 2$ and $n \geq 1$, we have $\log_c(U_n(c+1,c))=\log_c(c^n-1)-\log_c(c-1)$, which is equal to

$$n-1+\log_c\left(1-\frac{1}{c^n}\right)-\log_c\left(1-\frac{1}{c}\right).$$

Since $\log_c(1+y)=\frac{y}{\ln c}+o(y)\ (y\to 0)$,

$$\log_c\left(1-\frac{1}{c^n}\right)=-\frac{1}{c^n \ln c}+o\left(\frac{1}{c^n}\right)\ (n\to+\infty).$$

Further,

$$-\frac{1}{c-1}<\ln\left(1-\frac{1}{c}\right)<0.$$

Therefore, we deduce

$$-\frac{1}{(c-1)\ln c} < \log_c\left(1-\frac{1}{c}\right) < 0.$$

Now we can write, setting

$$\delta(c) = \left|\log_c\left(1-\frac{1}{c}\right)\right| = \log_c\left(1+\frac{1}{c-1}\right),$$

the approximation to $\log_c(U_n(c+1,c))$ holding for large n,

$$\begin{aligned}\log_c(U_n(c+1,c)) &= n-1+\log_c\left(1-\frac{1}{c^n}\right)-\log_c\left(1-\frac{1}{c}\right)\\ &= n-1+\delta(c)-\frac{1}{c^n\ln c}+o\left(\frac{1}{c^n}\right) \quad (n\to+\infty),\end{aligned}$$

where $0<\delta(c)<\frac{1}{(c-1)\ln c}$.

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AMS Classification Numbers: 11B39, 11A07

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