COMBINATORIAL MATRICES AND LINEAR RECURSIVE SEQUENCES

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1. INTRODUCTION

Various authors (see, e.g., [5], [7], [16], [17]) have studied number theoretic properties associated with the matrix $S(n)$, defined in effect by

$$S(n) = [s_{i,j}]_{n\times n}, \tag{1.1}$$

where

$$s_{i,j}(n) \equiv s_{i,j} = (-1)^{n-i}\binom{j-1}{n-i}p^{i+j-n-1}q^{n-i}, \tag{1.2}$$

where p, q are arbitrary integers. These properties have generally been in the context of second-order linear recursive sequences, particularly the Fibonacci numbers. We note that, for Horadam's generalized sequence $\{w_n\} = \{w_n(a, b; p, q)\}$ [13], we have the recurrence relation

$$w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2, \tag{1.3}$$

with initial conditions $w_0 = a$, $w_1 = b$. For the matrix S, we have the comparable partial recurrence relation

$$s_{i,j} = ps_{i,j-1} - qs_{i+1,j-1}. \tag{1.4}$$

We define the combinatorial matrix [2]: $S_{p,q}(n;2) = [|s_{i,j}(n)|]_{n\times n}$.

The purpose of this paper is to show how higher-order sequences arise quite naturally from $S(n)$ and to suggest problems for analogous further research arising out of further generalizations of the binomial coefficients. For notational purposes, we consider $S_{p,q}(n;r)$, where $S_{p,q}(n;2) = S(n)$ above, and for simplicity we take the absolute values of the numbers in the cells of each matrix.

2. PRELIMINARY OBSERVATIONS

We now have

$$S_{1,-1}(7;2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 & 5 & 15 \\ 0 & 0 & 0 & 1 & 4 & 10 & 20 \\ 0 & 0 & 1 & 3 & 6 & 10 & 15 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We observe that the falling diagonal sums are the Fibonacci numbers $\{1, 1, 2, 3, 5, 8, 13\}$ and the rising diagonal sums are the binomial coefficients $\{7, 21, 35, 21, 7, 1\}$. Similarly,

$$S_{2,-1}(7;2)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 & 1 & 10 & 60 \\ 0 & 0 & 0 & 1 & 8 & 40 & 160 \\ 0 & 0 & 1 & 6 & 24 & 80 & 240 \\ 0 & 1 & 4 & 12 & 32 & 80 & 192 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 \end{pmatrix}.$$

Other generalizations can be pursued. For instance,

$$S'_{2^k,-1}S_{2^k,-1}=S_{2^{k+1},-1}, \quad k\geq 0, \tag{2.1}$$

where

$$S'_{2^k,-1}=S_{2^k,-1}E$$

in which E is the elementary (self-inverse) matrix

$$E=[e_{i,j}]_{n\times n}$$

$$e_{i,j}=\begin{cases}1 & \text{if } j=n+1, \\ 0 & \text{otherwise.}\end{cases}$$

E is the unit matrix with rows reversed. It is used again in Section 5. An example of (2.1) when $k=1$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ 12 & 4 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 6 \\ 0 & 1 & 4 & 12 \\ 1 & 2 & 4 & 8 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 12 \\ 0 & 1 & 8 & 48 \\ 1 & 4 & 16 & 64 \end{pmatrix}.$$

The falling (from left to right) diagonal sums in these matrices are generalized Pell numbers, $\{P_n\}$, defined in turn by the second-order linear recurrence relations

$$P_n=2^k P_{n-1}+P_{n-2}, \quad n\geq 2,\ k\geq 0 \tag{2.2}$$

with initial conditions $P_0=0$, $P_1=1$. When $k=0,1$, we have the ordinary Fibonacci and Pell numbers, respectively.

In what follows, we use Bondarenko's notation $\binom{n}{m}_r$ for the number of different ways of distributing m objects among n cells where each cell may contain at most $r-1$ objects [3]:

$$\binom{n}{0}_r=\binom{n}{r-1}_r=1,$$

$$\binom{n}{m}_r=\binom{n}{(r-1)n-m}_r,$$

$$\binom{n}{m}_r=\begin{cases}0, & n<0, m<0, \text{ or } m>(r-1)n, \\ 1, & n=m=0.\end{cases}$$

$$\binom{n}{m}_r=\sum_{i=1}^{r}\binom{n-1}{m-i+1}_r.$$

3. THE A AND S MATRICES

We define the A and S matrices by

$$A(n;r)=\left[\binom{n-j}{j-i}_r\right] \tag{3.1}$$

and

$$S(n;r)=\left[\binom{j-1}{n-i}_r\right]. \tag{3.2}$$

For related developments, see [4], [8], [18]. As examples, we now look at $S(7;2)\equiv S_{1,-1}(7;2)$ and the associated matrix $A(7;2)$,

$$A(7;2)=\begin{pmatrix}1&5&6&1&0&0&0\\0&1&4&3&0&0&0\\0&0&1&3&1&0&0\\0&0&0&1&2&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}.$$

Then it is readily verified that

$$A(7;2)S(7;2)=S(7;3), \tag{3.3}$$

where

$$S(7;3)=\begin{pmatrix}0&0&0&1&10&45&141\\0&0&0&3&16&51&126\\0&0&1&6&19&45&90\\0&0&2&7&16&30&50\\0&1&3&6&10&15&21\\0&1&2&3&4&5&6\\1&1&1&1&1&1&1\end{pmatrix},$$

with falling diagonal sums $\{1,1,2,4,7,13,24\}$ which is a subset of the set of n-step self-avoiding walks on a Manhattan lattice, and the elements satisfy the linear third-order recurrence relation $u_n=u_{n-1}+u_{n-2}+u_{n-3}$, $n\geq 3$, with $u_0=0$, $u_1=1$, $u_2=1$ (see [21]). Next, let

$$A(7;3)=\begin{pmatrix}1&5&10&7&1&0&0\\0&1&4&6&2&0&0\\0&0&1&3&3&0&0\\0&0&0&1&2&1&0\\0&0&0&0&1&1&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}$$

and

$$S(7;4)=\begin{pmatrix}0&0&1&10&44&135&336\\0&0&2&12&40&101&216\\0&0&3&12&31&65&120\\0&1&4&10&20&35&56\\0&1&3&6&10&15&21\\0&1&2&3&4&5&6\\1&1&1&1&1&1&1\end{pmatrix}.$$

Then

$$A(7;3)S(7;2) = S(7;4). \tag{3.4}$$

More generally,

Theorem 1: $A(n;r)S(n;2) = S(n;r+1)$.

Proof:

$$\begin{aligned}
A(n;r)S(n;2) &= \left[\sum_{k=1}^{n} \binom{n-k}{k-i}_r \binom{j-1}{n-k}\right] \\
&= \left[\sum_{k=0}^{n} \binom{n-k}{k-i}_r \binom{j-1}{n-k}\right] && \text{from the definition of } \binom{n}{m}_r, \\
&= \left[\sum_{k=0}^{n} \binom{k}{n-k-i}_r \binom{j-1}{k}\right] && \text{reversing the order of summation,} \\
&= \left[\binom{j-1}{n-i}_{r+1}\right] && \text{from Equation (1.15) of [3],} \\
&= S(n;r+1). \;\square
\end{aligned}$$

The elements of S and A can be rearranged to form generalized Pascal triangles (see [19], [22], [25]). They can also be made into tetrahedrons with Pascal's triangle as one section (see [11], [12], [21]). Ericksen [9] has elaborated the principal properties of Bondarenko's coefficients in a pyramid.

4. RECURSIVE SEQUENCES

The rising diagonal sums associated with each of the r^{th} rows in the triangles of Section 3 yield the Fibonacci sequences and their generalizations; that is, the rising diagonals associated with the combined second rows yield the Fibonacci numbers. We can express this by the following theorem.

Theorem 2:

$$\sum_{k=0}^{\lfloor (r-1)n/r \rfloor} \binom{n-k}{k}_r = U_{n+1},$$

in which $\{U_n\}$ is the generalized Fibonacci sequence of arbitrary order r defined by the recurrence relation

$$U_n = \sum_{j=1}^{r} U_{n-j}, \quad n > 1,$$

with initial conditions $U_{-n} = 0$, $n = 0, 1, 2, \ldots, r-2$, $U_1 = 1$.

Proof: Consider

$$d_n = \sum_{k=0}^{n} \binom{n-k}{k}_r = \sum_{k=0}^{\lfloor (r-1)n/r \rfloor} \binom{n-k}{k}_r$$

from a consideration of the zero terms in the upper portion of the $\binom{n}{m}_r$ array. Then $d_0 = 1$, $d_n = 0$ for $n < 0$ and, for $n > 0$,

$$d_n = \sum_{k=0}^{n}\sum_{i=1}^{r}\binom{n-k-1}{k-i+1}_r = \sum_{i=1}^{r}\sum_{k=0}^{n}\binom{n-k-1}{k-i+1}_r$$

$$= \sum_{i=1}^{r}\sum_{k=-i+1}^{n-i+1}\binom{n-i-k}{k}_r \quad \text{changing the summation index to } j=k-i+1 \text{ then reverting to } k,$$

$$= \sum_{i=1}^{r}\sum_{k=0}^{n-i}\binom{n-i-k}{k}_r \quad \text{using the boundary conditions,}$$

$$= \sum_{i=1}^{r} d_{n-i}.$$

Thus, d_n satisfies the generalized Fibonacci recurrence relation of arbitrary order r with the given initial conditions. □

Basically, this theorem says that each element in the $\binom{n}{m}_r$ array is the sum of r elements above and to the left of it, and that r consecutive diagonals are needed to obtain all the terms required to form the elements of the next diagonal.

When $r = 2$, the theorem reduces to a familiar expression for the Fibonacci numbers, namely,

$$F_{n+1} = \sum_{m=0}^{\lfloor n/2 \rfloor}\binom{n-m}{m}, \tag{4.1}$$

and when $r = 3$, we get equation (4.1) of [21]:

$$U_{n+1} = \sum_{m=0}^{\lfloor n/2 \rfloor}\sum_{j=0}^{\lfloor n/3 \rfloor}\binom{n-m-j}{m+j}\binom{m+j}{j} = \sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n-k}{k}_3. \tag{4.2}$$

5. INVERSE MATRICES

The inverse matrices have some neat properties. For instance, for absolute values of the entries, we have

$$S^{-1} = ESE, \tag{5.1}$$

where E is the elementary matrix defined in Section 2. Of more interest is

$$A_{1,-1}^{-1}(7;2) = \begin{pmatrix} 1 & -5 & 14 & -28 & 42 & -42 & 0 \\ 0 & 1 & -4 & 9 & -14 & 14 & 0 \\ 0 & 0 & 1 & -3 & 5 & -5 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

The absolute values of the elements of the columns of A^{-1} can be rearranged to form the rows of Table 1. The row and column headed M refer to the corresponding sequence in Sloane and Plouffe [23].

The elements $a_{i,j}$ in Table 1 satisfy the partial recurrence relation

$$a_{i,j} = a_{i-1,j} + a_{i+1,j-1}, \quad i, j \geq 1,$$

with boundary conditions

$$a_{i,0}=1,\quad a_{0,j}=\frac{\binom{2j}{j}}{j+1}=c_j,$$

a Catalan number [14]. A general solution of this is given by

$$a_{i,j}=\frac{i+1}{1+j+1}\binom{2j+1}{j}.$$

TABLE 1. Elements of the Inverse Associated Matrix

$i\setminus j$	0	1	2	3	4	5	6		M
0	1	1	2	5	14	42	132		1459
1	1	2	5	14	42	132	429		1459
2	1	3	9	28	90	297	1001		2809
3	1	4	14	48	165	572	2002		3483
4	1	5	20	75	275	1001	3640		3904
5	1	6	27	110	429	1638	6188		4177
6	1	7	35	154	637	2548	9996		4413
M		1356	3841	4929	5277	–			

Note that the rising diagonals in Table 1 generate the Catalan numbers. The elements in Table 1 correspond to the number of two element lattice permutations, where the permutation represents a path through a lattice where the path does not cross a diagonal [6]. Since there are some intersections among the sequences in Table 1, a topic for further research could be to consider if these are the only intersections (cf. [24]).

Bondarenko's generalization of the binomial coefficient takes no account of the order across or within cells. Further research could accommodate this order and then apply these extensions to other combinatorial applications along the lines of the work of Letac and Takács [15] who, in effect, related the permutations associated with Bondarenko's $\binom{3}{m}_3$ to random walks along the edges of a dodecahedron or the connections of combinatorial matrices to planar networks [10]. Such research should lead to generalizations of the Fibonacci sequence which would be different from the $\{U_n\}$ discussed here and the standard generalizations of Philippou and his colleagues [20].

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