ON CHEBYSHEV POLYNOMIALS AND FIBONACCI NUMBERS*

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1. INTRODUCTION AND RESULTS

As usual, Chebyshev polynomials of the first and second kind, $T(x)=\{T_n(x)\}$ and $U(x)=\{U_n(x)\}$ $(n=0,1,2,\dots)$, are defined by the second-order linear recurrence sequences

$$T_{n+2}(x)=2xT_{n+1}(x)-T_n(x) \tag{1}$$

and

$$U_{n+2}(x)=2xU_{n+1}(x)-U_n(x) \tag{2}$$

for $n\geq 0$, $T_0(x)=1$, $T_1(x)=x$, $U_0(x)=1$, and $U_1(x)=2x$. These polynomials play a very important role in the study of the orthogonality of functions (see [1]), but regarding their arithmetical properties, we know very little at present. We do not even know whether there exists any relation between Chebyshev polynomials and some famous sequences. In this paper, we want to prove some identities involving Chebyshev polynomials, Lucas numbers, and Fibonacci numbers. For convenience, we let $T_n^{(k)}(x)$ and $U_n^{(k)}(x)$ denote the k^{th} derivatives of $T_n(x)$ and $U_n(x)$ with respect to x. Then we can use the generating functions of the sequences $T_n(x)$ and $U_n(x)$, and their partial derivatives, to prove the following three theorems.

Theorem 1: Let $U_n(x)$ be defined by (2). Then, for any positive integer k and nonnegative integer n, we have the identity

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n}\prod_{i=1}^{k+1}U_{a_i}(x)=\frac{1}{2^k\cdot k!}U_{n+k}^{(k)}(x),$$

where the summation is over all $k+1$-dimension nonnegative integer coordinates $(a_1,a_2,\dots,a_{k+1})$ such that $a_1+a_2+\cdots+a_{k+1}=n$.

Theorem 2: Under the conditions of Theorem 1, we have

$$\sum_{a_1+\cdots+a_{k+1}=n+2k+2}\prod_{i=1}^{k+1}(a_i+1)U_{a_i}(x)=\frac{1}{2^{2k+1}\cdot(2k+1)!}\sum_{h=0}^{k+1}(-1)^h\binom{k+1}{h}U_{n+4k+3-2h}^{(2k+1)}(x),$$

where $\binom{k}{h}=\frac{k!}{h!(k-h)!}$.

Theorem 3: Under the conditions of Theorem 1, we also have

$$\sum_{a_1+\cdots+a_{k+1}=n+k+1}\prod_{i=1}^{k+1}T_{a_i}(x)=\frac{1}{2^k\cdot k!}\sum_{h=0}^{k+1}(-x)^h\binom{k+1}{h}U_{n+2k+1-h}^{(k)}(x).$$

From these theorems, we may immediately deduce the following corollaries.

Corollary 1: Let F_n be the n^{th} Fibonacci number. Then, for any positive integer k and nonnegative integer n, we have the identities:

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$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} F_{a_1+1}\cdot F_{a_2+1}\cdot\cdots\cdot F_{a_{k+1}+1}=\frac{(-i)^n}{2^k\cdot k!}U_{n+k}^{(k)}\left(\frac{i}{2}\right),$$

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} F_{2(a_1+1)}\cdot F_{2(a_2+1)}\cdot\cdots\cdot F_{2(a_{k+1}+1)}=\frac{(-1)^n}{2^k\cdot k!}U_{n+k}^{(k)}\left(\frac{-3}{2}\right),$$

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} F_{3(a_1+1)}\cdot F_{3(a_2+1)}\cdot\cdots\cdot F_{3(a_{k+1}+1)}=\frac{2i^n}{k!}U_{n+k}^{(k)}(-2i),$$

where $i^2=-1$. In particular, for $k=2$, we have the identities:

$$\sum_{a+b+c=n} F_{a+1}\cdot F_{b+1}\cdot F_{c+1}=\frac{1}{50}[(n+2)(5n+17)F_{n+3}-6(n+3)F_{n+2}],$$

$$\sum_{a+b+c=n} F_{2(a+1)}\cdot F_{2(b+1)}\cdot F_{2(c+1)}=\frac{1}{50}[18(n+3)F_{2n+4}+(n+2)(5n-7)F_{2n+6}],$$

$$\sum_{a+b+c=n} F_{3(a+1)}\cdot F_{3(b+1)}\cdot F_{3(c+1)}=\frac{1}{50}[(n+2)(5n+8)F_{3n+9}-6(n+3)F_{3n+6}].$$

Corollary 2: Under the conditions of Corollary 1, we have:

$$\sum_{a_1+\cdots+a_{k+1}=n+2k+2} (a_1+1)\cdots(a_{k+1}+1)\cdot F_{a_1+1}\cdots F_{a_{k+1}+1}$$
$$=\frac{(-i)^{n+2k+2}}{2^{2k+1}\cdot(2k+1)!}\sum_{h=0}^{k+1}(-1)^h\binom{k+1}{h}U_{n+4k+3-2h}^{(2k+1)}\left(\frac{i}{2}\right),$$

$$\sum_{a_1+\cdots+a_{k+1}=n+2k+2} (a_1+1)\cdots(a_{k+1}+1)\cdot F_{2(a_1+1)}\cdots F_{2(a_{k+1}+1)}$$
$$=\frac{(-1)^n}{2^{2k+1}\cdot(2k+1)!}\sum_{h=0}^{k+1}(-1)^h\binom{k+1}{h}U_{n+4k+3-2h}^{(2k+1)}\left(\frac{-3}{2}\right),$$

$$\sum_{a_1+\cdots+a_{k+1}=n+2k+2} (a_1+1)\cdots(a_{k+1}+1)\cdot F_{3(a_1+1)}\cdots F_{3(a_{k+1}+1)}$$
$$=\frac{i^{n+2k+2}}{2^k\cdot(2k+1)!}\sum_{h=0}^{k+1}(-1)^h\binom{k+1}{h}U_{n+4k+3-2h}^{(2k+1)}(-2i).$$

Corollary 3: Let L_n be the n^{th} Lucas numbers. Then, for any positive integer k and nonnegative integer n, we have the identities:

$$\sum_{a_1+\cdots+a_{k+1}=n+k+1} L_{a_1}\cdot L_{a_2}\cdot\cdots\cdot L_{a_{k+1}}=\frac{(-i)^{n+k+1}}{2^{-1}\cdot k!}\sum_{h=0}^{k+1}\left(\frac{-i}{2}\right)^h\binom{k+1}{h}U_{n+2k+1-h}^{(k)}\left(\frac{i}{2}\right),$$

$$\sum_{a_1+\cdots+a_{k+1}=n+k+1} L_{2a_1}\cdot L_{2a_2}\cdot\cdots\cdot L_{2a_{k+1}}=\frac{(-i)^{n+k+1}}{2^{-1}\cdot k!}\sum_{h=0}^{k+1}\left(\frac{3}{2}\right)^h\binom{k+1}{h}U_{n+2k+1-h}^{(k)}\left(\frac{-3}{2}\right),$$

$$\sum_{a_1+\cdots+a_{k+1}=n+k+1} L_{3a_1}\cdot L_{3a_2}\cdot\cdots\cdot L_{3a_{k+1}}=\frac{i^{n+k+1}}{2^{-1}\cdot k!}\sum_{h=0}^{k+1}(2i)^h\binom{k+1}{h}U_{n+2k+1-h}^{(k)}(-2i),$$

where $i^2=-1$. In particular, for $k=2$, we have the identities:

$$\sum_{a+b+c=n+3} L_a \cdot L_b \cdot L_c = \frac{n+5}{2}[(n+10)F_{n+3} + 2(n+7)F_{n+2}],$$

$$\sum_{a+b+c=n+3} L_{2a} \cdot L_{2b} \cdot L_{2c} = \frac{n+5}{2}[3(n+10)F_{2n+5} + (n+16)F_{2n+4}],$$

$$\sum_{a+b+c=n+3} L_{3a} \cdot L_{3b} \cdot L_{3c} = \frac{n+5}{2}[4(n+10)F_{3n+7} + 3(n+9)F_{3n+6}].$$

Corollary 4: For any nonnegative integer n, we have the congruence

$$(n+2)(5n+8)F_{3n+9} \equiv 6(n+3)F_{3n+6} \mod 400.$$

These corollaries are generalizations of [2].

2. PROOF OF THE THEOREMS

In this section we shall complete the proofs of the theorems. First, note that (see [1], (2.1.1))

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right]$$

and

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right],$$

so we can easily deduce that the generating function of $T(x)$ and $U(x)$ are

$$G(t,x) = \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{+\infty} T_n(x)\cdot t^n \tag{3}$$

and

$$F(x,t) = \frac{1}{1-2xt+t^2} = \sum_{n=0}^{+\infty} U_n(x)\cdot t^n, \tag{4}$$

respectively. Then from (4) we have

$$\begin{aligned}
\frac{\partial F(t,x)}{\partial x} &= \frac{2t}{(1-2xt+t^2)^2} = \sum_{n=0}^{\infty} U_{n+1}^{(1)}(x)\cdot t^{n+1},\\
\frac{\partial^2 F(t,x)}{\partial x^2} &= \frac{2!\cdot(2t)^2}{(1-2xt+t^2)^3} = \sum_{n=0}^{\infty} U_{n+2}^{(2)}(x)\cdot t^{n+2},\\
&\cdots\\
\frac{\partial^k F(t,x)}{\partial x^k} &= \frac{k!\cdot(2t)^k}{(1-2xt+t^2)^{k+1}} = \sum_{n=0}^{\infty} U_{n+k}^{(k)}(x)\cdot t^{n+k},
\end{aligned} \tag{5}$$

where we have used the fact that $U_n(x)$ is a polynomial of degree n.

Therefore, from (5) we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{a_1+\cdots+a_{k+1}=n} U_{a_1}(x)\cdot U_{a_2}(x)\cdot\cdots\cdot U_{a_{k+1}}(x)\right)\cdot t^n = \left(\sum_{n=0}^{\infty} U_n(x)\cdot t^n\right)^{k+1}\\
&= \frac{1}{(1-2xt+t^2)^{k+1}} = \frac{1}{k!(2t)^k}\frac{\partial^k F(t,x)}{\partial x^k} = \frac{1}{2^k\cdot k!}\sum_{n=0}^{\infty} U_{n+k}^{(k)}(x)\cdot t^n.
\end{aligned} \tag{6}$$

Equating the coefficients of t^n on both sides of equation (6), we obtain the identity

$$\sum_{a_1+a_2+\cdots a_{k+1}=n} U_{a_1}(x)\cdot U_{a_2}(x)\cdot\cdots\cdot U_{a_{k+1}}(x)=\frac{1}{2^k\cdot k!}\cdot U_{n+k}^{(k)}(x).$$

This proves Theorem 1.

Now we prove Theorem 3. Multiplying both sides of (5) by $(1-xt)^{k+1}$ gives

$$\frac{(1-xt)^{k+1}}{(1-2xt+t^2)^{k+1}}=\frac{1}{2^k\cdot k!}\sum_{n=0}^{\infty}U_{n+k}^{(k)}(x)\cdot t^n(1-xt)^{k+1}. \tag{7}$$

Note that

$$(1-xt)^{k+1}=\sum_{h=0}^{k+1}(-x)^h t^h\binom{k+1}{h}.$$

Comparing the coefficients of t^{n+k+1} on both sides of equation (7), we obtain Theorem 3.

To prove Theorem 2, we note that $\frac{d(T_n(x))}{dx}=nU_{n-1}(x)$ and

$$\frac{\partial G(t,x)}{\partial x}=\frac{t-t^3}{(1-2xt+t^2)^2}=\sum_{n=0}^{\infty}T_{n+1}^{(1)}(x)\cdot t^{n+1}$$

or

$$\frac{1-t^2}{(1-2xt+t^2)^2}=\sum_{n=0}^{\infty}(n+1)U_n(x)\cdot t^n. \tag{8}$$

Taking $k=2m+1$ in (5), then multiplying by $(1-t^2)^{m+1}$ on both sides of (5), we can also get

$$\frac{(1-t^2)^{m+1}}{(1-2xt+t^2)^{2m+2}}=\frac{1}{2^{2m+1}\cdot(2m+1)!}\sum_{n=0}^{\infty}U_{n+2m+1}^{(2m+1)}(x)\cdot t^n(1-t^2)^{m+1}. \tag{9}$$

Combining (8) and (9), we may immediately obtain the identity

$$\sum_{a_1+\cdots+a_{m+1}=n+2m+2}(a_1+1)\cdots(a_{m+1}+1)\cdot U_{a_1}(x)\ldots U_{a_{m+1}}(x)$$

$$=\frac{1}{2^{2m+1}\cdot(2m+1)!}\sum_{h=0}^{m+1}(-1)^h\binom{m+1}{h}U_{n+4m+3-2h}^{(2m+1)}(x).$$

This completes the proof of Theorem 2.

Proof of the Corollaries: Taking $x=\frac{i}{2}$, $\frac{-3}{2}$, and $-2i$ in Theorems 1-3, respectively, and noting that

$$U_n\left(\frac{i}{2}\right)=i^nF_{n+1},\ U_n\left(\frac{-3}{2}\right)=(-1)^nF_{2(n+1)},\ U_n(-2i)=\frac{(-i)^n}{2}F_{3(n+1)},$$

$$T_n\left(\frac{i}{2}\right)=\frac{i^n}{2}L_n,\ T_n\left(\frac{-3}{2}\right)=\frac{(-1)^n}{2}L_{2n},\ T_n(-2i)=\frac{(-i)^n}{2}L_{3n},$$

$$F_{n+2}=F_{n+1}+F_n,$$

$$(1-x^2)U_n'(x)=(n+1)U_{n-1}(x)-nxU_n(x),$$

and

$$(1-x^2)U_n''(x)=3xU_n'(x)-n(n+2)U_n(x),$$

we may immediately deduce Corollaries 1-3. Corollary 4 follows from Corollary 1 and the fact that $2|F_{3(a+1)}$ for all integers $a \geq 0$.

Remark: For any positive integer $m \geq 4$, using our theorems, we can also give an exact calculating formula for the general sums

$$\sum_{a_1+\cdots+a_k=n} \prod_{i=1}^{k} F_{m(a_i+1)} \quad \text{and} \quad \sum_{a_1+\cdots+a_k=n+k} \prod_{i=1}^{k} L_{ma_i},$$

but in these cases the computations are more complex.

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