ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

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PROBLEMS PROPOSED IN THIS ISSUE

H-593 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Let $p>5$ be a prime. Prove the congruence

$$2\sum_{k=0}^{[(p-5)/10]}\frac{(-1)^k}{2k+1}\equiv(-1)^{(p-1)/2}\frac{2^{p-1}-L_p}{p}\pmod{p}.$$

H-594 ***Proposed by Mario Catalani, University of Torino, Torino, Italy***

Consider the generalized Fibonacci and Lucas polynomials:

$$F_{n+1}(x,y)=xF_n(x,y)+yF_{n-1}(x,y),\ F_0(x,y)=0,\ F_1(x,y)=1;$$
$$L_{n+1}(x,y)=xL_n(x,y)+yL_{n-1}(x,y),\ L_0(x,y)=2,\ L_1(x,y)=x.$$

Assume $y\neq 0,\ 2x^2-y\neq 0$. We will write F_n and L_n for $F_n(x,y)$ and $L_n(x,y)$, respectively. Show that:

1. $$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}x^ky^{-2k}F_{3k}=\frac{xF_{2n+1}-yF_{2n}+(-x)^{n+2}F_n+(-x)^{n+1}yF_{n-1}}{y^n(2x^2-y)};$$

2. $$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}x^ky^{-2k}L_{3k}=\frac{xL_{2n+1}-yL_{2n}+(-x)^{n+2}L_n+(-x)^{n+1}yL_{n-1}}{y^n(2x^2-y)}.$$

H-595 ***Proposed by José Díaz-Barrero & Juan Egozcue, Barcelona, Spain***

Let ℓ, n be positive integers. Prove that

$$\sum_{k=0}^{n}\binom{k+\ell+1}{k+1}\left\{\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}P_n^{j-k-1}\right\}=P_n^{\ell+1}-1,$$

where P_n is the n^{th} Pell number, i.e., $P_0=0$, $P_1=1$, and $P_{n+2}=2P_{n+1}+P_n$ for $n\geq 2$.

SOLUTIONS

A Convoluted Problem

H-583 *Proposed by N. Gauthier, Royal Military College of Canada (Vol. 40, no. 2, May 2002)*

A Theorem on Generalized Fibonacci Convolutions

This is a generalization of Problem B-858 by W. Lang (*The Fibonacci Quarterly* **36.3**, 1998).

Let $n \geq 0$, a, b be integers; also let A, B be arbitrary yet known real numbers and consider the generalized Fibonacci sequence $\{G_n \equiv A\alpha^n + B\beta^n\}_{n=-\infty}^{\infty}$, where

$$\alpha = \frac{1}{2}[1+\sqrt{5}],\ \beta = \frac{1}{2}[1-\sqrt{5}].$$

For m a nonnegative integer, prove the following generalized convolution theorem for the sequences $\{(a+n)^m\}_{n=-\infty}^{\infty}$ and $\{G_n\}_{n=-\infty}^{\infty}$,

$$\sum_{k=0}^{n}(a+k)^m G_{b-a-k} = \sum_{l=0}^{m} l![c_l^m(a)G_{b-a+l+1} - c_l^m(a+n+1)G_{b-a-n+1+l}],$$

where the set of coefficients $\{c_l^m(v);\ 0 \leq m;\ 0 \leq l \leq m;\ v = a \text{ or } a+n+1\}$ satisfies the following second-order linear recurrence relation

$$c_l^{m+1}(v) = (v+l)c_l^m(v) + c_{l-1}^m(v);\ c_{l=0}^{m=0}(v) = 1, c_{l=0}^{m=1}(v) = v, c_{l=1}^{m=1}(v) = 1$$

with the understanding that $c_{-1}^m(v) \equiv 0$ and that $c_{m+1}^m(v) \equiv 0$.

Prob. B-858 follows as a special case if one sets $a = 0$, $m = 1$, $b = n$, and $A = -B = (\alpha-\beta)^{-1}$ in the above theorem. Indeed, one then gets that

$$G_n = F_n,\ c_0^1(0) = 0,\ c_1^1(0) = 1,\ c_0^1(n+1), \text{ and } c_1^1(n+1) = 1$$

and the result follows directly.

Solution by Paul S. Bruckman, Berkeley, CA

For typographical clarity, we change the summation variable "l" to "j" and we also change the notation "$c_j^m(x)$" to "$c(x; j, m)$". We also note that there is a misprint in the statement of the problem. The correct expression in the right member of the statement of the problem (as modified by the indicated changes in notation) is as follows:

$$\sum_{j=0}^{m} j![c(\alpha; j, m)\, G_{b-a+j+2} - c(\alpha+n+1; j, m)\, G_{b-a-n+j+1}].$$

We employ the standard finite difference operators Δ and $E \equiv 1+\Delta$, where the operand is x. We first demonstrate the following result:

$$c(x; j, m) = \Delta^j / j! = (x^m). \tag{1}$$

Proof of (1): Let $d(x; j, m) = \Delta^j / j!(x^m)$, $0 \leq j \leq m$ for all real x. Clearly, $d(x; j, m)$ is a polynomial in x. Note that $d(x; 0, m) = x^m$. Also, $d(x; m, m) = 1$ for all m and x, and $d(x; 0, 1) = x$. Thus, the boundary conditions satisfied by the $c(x; j, m)$ are also satisfied by the $d(x; j, m)$.

Next, note that

$$
\begin{aligned}
d(x;j,m+1) &= \Delta^j/j!(x^{m+1}) = (1/j!)\sum_{k=0}^{j} {}_jC_k(-1)^k(x+j-k)^{m+1} \\
&= \{(x+j)/j!\}\sum_{k=0}^{j} {}_jC_k(-1)^k(x+j-k)^m - \{j/j!\}\sum_{k=1}^{j} {}_{j-1}C_{k-1}(-1)^k(x+j-k)^m \\
&= \{(x+j)/j!\}\sum_{k=0}^{j} {}_jC_k(-1)^k(x+j-k)^m + \{1/(j-1)!\}\sum_{k=0}^{j-1} {}_{j-1}C_k(-1)^k(x+j-1-k)^m \\
&= (x+j)\{\Delta^j/j!\}(x^m) + \{\Delta^{j-1}/(j-1)!\}(x^m) = (x+j)d(x;j,m) + d(x;j-1,m).
\end{aligned}
$$

This is the same recurrence as the one satisfied by the $c(x;j,m)$. Since the two-dimensional sequences $c(x;j,m)$ and $d(x;j,m)$ satisfy the same recurrence and have the same boundary conditions, they must be identical. This establishes (1). □

Therefore, the left member of the putative identity (denoted as $\mathfrak{L}$) is transformed as follows:

$$
\begin{aligned}
\mathfrak{L} &= \sum_{k=0}^{n} E^k(x^m)G_{b-a-k}\Big|_{x=\alpha} = \sum_{k=0}^{n} E^k(x^m)\{A\alpha^{b-a-k} + B\beta^{b-a-k}\}\Big|_{x=\alpha} \\
&= [A\alpha^{b-a}\{(E/\alpha)^{n+1}-1\}/(E/\alpha-1)\} + B\beta^{b-a}\{(E/\beta)^{n+1}-1\}/(E/\beta-1)\}](x^m)\Big|_{x=\alpha}
\end{aligned}
$$

or

$$
\mathfrak{L} = [A\alpha^{b-a-n}\{E^{n+1}-\alpha^{n+1})/(E-\alpha)\} + B\beta^{b-a-n}\{(E^{n+1}-\beta^{n+1})/(E-\beta)\}](x^m)\Big|_{x=\alpha}. \tag{2}
$$

On the other hand, if $\mathfrak{R}$ represents the right member of the (corrected) putative identity, then

$$
\mathfrak{R} = \sum_{k=0}^{m} \Delta^k(x^m)\{A\alpha^{b-a+2+k} + B\beta^{b-a+2+k}\}\Big|_{x=\alpha} - \sum_{k=0}^{m} \Delta^k(x^m)\{A\alpha^{b-a+1-n+k} + B\beta^{b-a+1-n+k}\}\Big|_{x=\alpha+n+1}
$$

$$
\begin{aligned}
\mathfrak{R} = {} & [A\alpha^{b-a+2}\{(\Delta\alpha)^{m+1}-1\}/(\Delta\alpha-1) + B\beta^{b-a+2}\{(\Delta\beta)^{m+1}-1\}/(\Delta\beta-1)\}](x^m)\Big|_{x=\alpha} \\
& - [A\alpha^{b-a+1-n}\{(\Delta\alpha)^{m+1}-1\}/(\Delta\alpha-1) + B\beta^{b-a+1-n}\{(\Delta\beta)^{m+1}-1\}/(\Delta\beta-1)](x^m)\Big|_{x=\alpha+n+1}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R} = {} & [A\alpha^{b-a+1-n}\{(\Delta\alpha)^{m+1}-1\}/(\Delta\alpha-1)(\alpha^{n+1}-E^{n+1}) \\
& + B\beta^{b-a+1-n}\{(\Delta\beta)^{m+1}-1\}/(\Delta\beta-1)(\beta^{n+1}-E^{n+1})](x^m)\Big|_{x=\alpha}.
\end{aligned}
$$

Now note that $\Delta^{m+1}(x^m)=0$. Also, $\Delta\alpha-1=(E-1)\alpha-1=E\alpha-\alpha^2$, and $\Delta\beta-1=(E-1)\beta-1=E\beta-\beta^2$. Therefore, we see that

$$
\mathfrak{R} = A\alpha^{b-a-n}\{(E^{n+1}-\alpha^{n+1})/(E-\alpha)\} + B\beta^{b-a-n}\{(E^{n+1}-\beta^{n+1})/(E-\beta)\}(x^m)]\Big|_{x=\alpha}. \tag{3}
$$

Comparison of (2) and (3) shows that $\mathfrak{L}=\mathfrak{R}$. Q.E.D.

Also solved by the proposer.

Find Your Identity

<u>H-584</u> ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 40, no. 2, May 2002)

Prove the following identity:

$$
\begin{aligned}
&(F_{n+4}+L_{n+3})^5 + (F_n+L_{n+1})^5 + (2F_{n+1}+L_{n+2})^5 \\
&= (2F_{n+3}+L_{n+2})^5 + (F_{n+2})^5 + (5F_{n+2})^5 + 1920F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}.
\end{aligned}
$$

Solution by the proposer

We begin with the following identity:

$$\begin{aligned}&(a+b+c+d-e)^5+(a+b+c-d+e)^5+(a+b-c+d+e)^5+(a-b+c+d+e)^5\\&+(-a+b+c-d+e)^5+(a+b+c-d-e)^5+(a+b-c+d-e)^5+(a-b+c+d-e)^5\\&+(-a+b+c+d-e)^5+(a+b-c-d+e)^5+(a-b+c-d+e)^5+(-a+b+c-d+e)^5 \qquad (*)\\&+(a-b-c+d+e)^5+(-a-b-c+d+e)^5+(-a+b+c+d+e)^5+(a+b+c+d+e)^5\\&=1920abcde.\end{aligned}$$

We replace a, b, c, d, and e by x_1, x_2, x_3, x_4, and x_5, respectively. We may prove $(*)$ (as thus modified) by expanding

$$(x_1+x_2+x_3+x_4+x_5)^5=\sigma_5+5\sigma_{14}+10\sigma_{23}+20\sigma_{113}+30\sigma_{122}+60\sigma_{1112}+120\sigma_{11111},$$

where $\sigma_{abc}=\Sigma u^a v^b w^c$, for example (with u, v, and w representing the x_i's over all possible permutations), with similar definitions for other quantities. Then we note that in the sum of the 16 terms indicated in $(*)$, the terms involving $(x_1)^5$ vanish, since their coefficient is $+1$ 8 times and -1 8 times. The terms involving $(x_1)^4$ also vanish, since their coefficients are x_i twice and $-x_i$ twice (for each $i=2,3,4,$ or 5). The terms involving $(x_1)^3$ have two kinds of coefficients: $(x_i)^2$ and $-(x_i)^2$; also, $x_i x_j$ and $-x_i x_j$, where i or $j=2,3,4,$ or 5, $i\neq j$. In either case, each sign occurs an equal number of times, and so the term vanishes. The *remaining* terms involving $(x_1)^2$ have two kinds of coefficients: $x_i(x_j)^2$ and $-x_i(x_j)^2$; also $x_i x_j x_k$ and $-x_i x_j x_k$. Here, $i, j,$ or $k=2,3,4,$ or 5, with $i, j,$ and k distinct. In either case, the positive terms again cancel the negative ones, and so the terms involving $(x_1)^2$ all vanish. Finally, the *remaining* terms involving the first powers x_1 have coefficients $x_2x_3x_4x_5$ or $-x_2x_3x_4x_5$ for each of the 16 terms, but are such that the total term is always positive. Therefore, the total coefficient of the product $x_1x_2x_3x_4x_5$ is $16*120=1920$. By symmetry, the sum is therefore equal to $1920x_1x_2x_3x_4x_5$. Thus, $(*)$ is proved.

In particular, if we set $a=F_n$, $b=F_{n+1}$, $c=F_{n+2}$, $d=F_{n+3}$, and $e=F_{n+4}$, we obtain (after some simplification) the indicated result.

Also solved by K. Davenport, L. A. G. Dresel, O. Furdui, N. Tuglu, and H. Civciv.

A *D*-Sequence

H-585 ***Proposed by Herrmann Ernst, Siegburg, Germany***
(Vol. 40, no. 4, August 2002)

Let (d_n) denote a sequence of positive integers d_n with $d_1\geq 3$ and $d_{n+1}-d_n\geq 1$, $n=1,2,\ldots$. We introduce the following sets of sequences (d_n):

$$A=\left\{(d_n)\colon \sum_{k=1}^{\infty}\frac{1}{F_{d_k}}\leq 1\right\};$$

$$B=\left\{(d_n)\colon \frac{1}{F_{d_n}}<\sum_{k=n}^{\infty}\frac{1}{F_{d_k}}<\frac{1}{F_{d_n-1}} \text{ for all } n\in N\right\};$$

$$C=\left\{(d_n)\colon 0\leq\frac{1}{F_{d_n-1}}-\frac{1}{F_{d_n}}-\frac{1}{F_{d_{n+1}-1}} \text{ for all } n\in N\right\}.$$

Show that:

(a) there is a bijection $f:]0,1] \to B$, $f(x) = (d_n(x))_{n=1}^{\infty}$;

(b) B is a subset of A with $A \setminus B \neq \emptyset$;

(c) C is a subset of B with $B \setminus C \neq \emptyset$.

Solution by Paul S. Bruckman, Berkeley, CA

A sequence $(d_n)_{n=1}^{\infty}$ of positive integers is called a <u>D-sequence</u> iff $d_1 \geq 3$ and $d_{n+1} - d_n \geq 1$, $n \in N$. Let Δ denote the set of all D-sequences. Also, for typographical convenience, we write $F(k)$ for F_k. We also write $S_{n,M} = \Sigma_{k=n}^{M} 1/F(d_k)$, and $S_n = S_{n,\infty}$ for all $(d_n) \in \Delta$. For a given $\delta = (d_n) \in \Delta$, we may also write $S_n(\delta) = S_n$. We may characterize A, B, and C as follows:

$A = \{\delta = (d_n) \in \Delta : S_1(\delta) \geq 1\}$;

$B = \{\delta = (d_n) \in \Delta : 1/F(d_n) < S_n(\delta) \leq 1/F(d_n - 1) \text{ for all } n \in N\}$;

$C = \{\delta = (d_n) \in \Delta : 0 \leq 1/F(d_n - 1) - 1/F(d_n) - 1/F(d_{n+1} - 1) \text{ for all } n \in N\}$.

Note the slight modification in the definition of B ("$\leq$" instead of "$<$" in the second inequality defining B.

Proof of (a): Suppose $x_1 \in (0,1)$ is given. Then there exists $d_1 \in N$, $d_1 \geq 3$, such that $1/F(d_1) < x_1 \leq 1/F(d_1 - 1)$. Let $x_2 = x_1 - 1/F(d_1)$. Note that $0 < x_2 < x_1 < 1$. We continue in this fashion; generally, we define the sequence (x_n) as follows: $x_{n+1} = x_n - 1/F(d_n)$, $1/F(d_n) < x_n \leq 1/F(d_n - 1)$, $d_{n+1} > d_n$, $n \in N$. Note that (x_n) is a decreasing sequence, bounded below by zero. Since $d_n \to \infty$ as $n \to \infty$, we see that x_n is arbitrarily small. Hence, $\lim_{n\to\infty} x_n = 0$. By iteration, $x_1 = 1/F(d_1) + x_2 = 1/F(d_1) + 1/F(d_2) + x_3 = \cdots = S_{1,M} + x_{M+1}$ for all $M \in N$. Allowing $M \to \infty$, we deduce that $x_1 = S_1$, where the D-sequence (d_n) is uniquely determined by the construction indicated above. Note, however, that the maximum value of S_1 over the domain Δ is $\Sigma_{n=3}^{\infty} 1/F_n = \sigma$, say, where $\sigma \approx 1.3599$. In other words, there is *not* a one-to-one correspondence between $(0,1)$ and Δ, the set of all possible D-sequences. There are sequences $\delta \in \Delta$ such that $S_1(\delta) \geq 1$. We may use the same construction as before, if $1 \leq x_1 \leq \sigma$. For example,

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{34} + \frac{1}{89} + \frac{1}{987} + \frac{1}{196418} + \frac{1}{2178309} + \cdots$$
$$= \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_6} + \frac{1}{F_9} + \frac{1}{F_{11}} + \frac{1}{F_{16}} + \frac{1}{F_{27}} + \frac{1}{F_{32}} + \cdots,$$

corresponding to the D-sequence $\delta_1 = (3, 4, 6, 9, 11, 16, 27, 32, \ldots)$, such that $S_1(\delta_1) = 1$.

We are to establish that if $x_1 \in (0,1)$ then there exists a unique $\delta \in B$ such that $S_n(\delta) \leq 1/F(d_n - 1)$ for all $n \in N$; the other condition for $\delta \in B$, namely that $1/F(d_n) < S_n(\delta)$ for all $n \in N$, is automatically satisfied. We already know how to effect the construction of the unique $\delta \in \Delta$ such that $x_1 = S_1(\delta)$. It only remains to show that, for such δ, $S_n(\delta) \leq 1/F(d_n - 1)$ for all $n \in N$. Note that $x_1 = S_{1,n-1}(\delta) + x_n = x_1 - S_n(\delta) + x_n$ for all $n \in N$. This implies that $x_n = S_n(\delta)$ for all $n \in N$. By our construction, $x_n = S_n(\delta) \leq 1/F(d_n - 1)$ for all $n \in N$. This completes the proof of part (a).

Note: Although, for a given $x_1 \in (0,1)$, there exists a unique $\delta \in B$ corresponding to x_1 (as provided by our construction, and such that $S_1(\delta) = x_1$), there may be other $\delta \in \Delta \setminus B$, say δ', such that $S_1(\delta') = x_1$. An illustration of this is provided by $x_1 = \rho = 1/F_4 + 1/F_6 + 1/F_8 + \cdots \approx 0.5354$. Clearly, this is generated by the sequence $\delta' = \delta'(\rho) = (4, 6, 8, \ldots)$, which is an element of Δ. However, it is easily verified that δ' is not an element of B, since $1/F(d_1 - 1) = 1/F(3) = 1/2 < S_1(\delta') = \rho = 0.5354$. Our construction, however, yields the alternative sequence $\delta = \delta(\rho) = (3, 9, 13, 15, 24, 27, 31, 35, 37, 39, 42, 49, \ldots)$, which also has $S_1(\delta) = \rho = 0.5354$ and is, moreover, an element of B (this is true by the nature of the construction).

Proof of (b): Suppose $\delta \in B$. Then $S_1(\delta) \le 1/F(d_1 - 1) \le 1/F(2) = 1$; hence, $\delta \in A$. Thus, $B \subseteq A$.

As we have seen, $\delta' \in \Delta \setminus B$, where $\delta' = (4, 6, 8, \ldots) = (2n)_{n=2}^{\infty}$, but $S_1(\delta') = \rho \approx 0.5354 < 1$, so $\delta_1 \in A$. Hence, $\delta' \in A \setminus B$ and $A \setminus B \neq \varnothing$.

Proof of (c): Suppose $\delta \in C$. Then, for all $n \in N$, $1/F(d_n - 1) \ge 1/F(d_n) + 1/F(d_{n+1} - 1)$. By iteration, $1/F(d_n - 1) \ge 1/F(d_n) + 1/F(d_{n+1}) + \cdots + 1/F(d_M) + 1/F(d_{M+1} - 1)$ for all M, n with $M \ge n \ge 1$. Thus, $1/F(d_n - 1) > S_{n,M}(\delta)$ for all such M, n. Allowing $M \to \infty$, it follows that $1/F(d_n - 1) \ge S_n(\delta)$ for all $n \in N$. Therefore, $\delta \in B$, which shows that $C \subseteq B$.

We display an example of a sequence $\delta'' \in B \setminus C$.

We let $\delta'' = (6, 8, 10, 12, 15, 18, 20, 22, 24, 29, \ldots)$ represent the element of B determined by our construction, such that $S_1(\delta'') = 0.2$. By definition, $\delta'' \in B$. However,

$$\frac{1}{F(d_1 - 1)} - \frac{1}{F(d_1)} - \frac{1}{F(d_2 - 1)} = \frac{1}{F(5)} - \frac{1}{F(6)} - \frac{1}{F(7)} = \frac{1}{5} - \frac{1}{8} - \frac{1}{13} = \frac{-1}{520} < 0,$$

which shows that $\delta'' \notin C$; hence, $\delta'' \in B \setminus C$ and $B \setminus C \neq \varnothing$. This completes the proof of part (c).

Note: More generally, $1/F(2n-1) - 1/F(2n) - 1/F(2n+1) = -1/F(2n-1)F(2n)F(2n+1) < 0$, after simplification. Thus, given $\delta = (d_n) \in \Delta$ with $d_k = 2n$ and $d_{k+1} = 2n+2$, say, then $\delta \notin C$; i.e., if $d_k \in \delta \in C$ and d_k is even, then $d_{k+1} - d_k \ge 3$.

Also solved by the proposer.

Note: Problem **H-582** (proposed by Ernst Herrman and solved by Paul S. Bruckman) will appear in the May 2003 issue of this quarterly.

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