ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2003. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-951 ***Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA***

The sequence $\langle u_n \rangle$ is defined by the recurrence

$$u_{n+1} = \frac{3u_n + 1}{5u_n + 3}$$

with the initial condition $u_1 = 1$. Express u_n in terms of Fibonacci and/or Lucas numbers.

B-952 ***Proposed by Scott H. Brown, Auburn University, Montgomery, AL***

Show that

$$10F_{10n-5} + 12F_{10n-10} + F_{10n-15} = 25F_{2n}^5 + 25F_{2n}^3 + 5F_{2n}$$

for all integers $n \geq 2$.

B-953 ***Proposed by Harvey J. Hindin, Huntington Station, NY***

Show that

$$(F_n)^4 + (F_{n+1})^4 + (F_{n+2})^4$$

is never a perfect square. Similarly, show that

$$(qW_n)^4 + (pW_{n+1})^4 + (W_{n+2})^4$$

is never a perfect square, where W_n is defined for all integers n by $W_n = pW_{n-1} - qW_{n-2}$ and where $W_0 = a$ and $W_1 = b$.

B-954 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Let n be a nonnegative integer. Show that

$$\sqrt{(\sqrt{5}+2)(\sqrt{5}F_{2n+1}-2)} = L_{2\lfloor n/2\rfloor+1} + \sqrt{5}F_{2\lceil n/2\rceil},$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor- and ceiling-function, respectively.

B-955 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI***

Prove that

$$1 < \frac{F_{2n}}{\sqrt{1+F_{2n}^2}} + \frac{1}{\sqrt{1+F_{2n+1}^2}} + \frac{1}{\sqrt{1+F_{2n+2}^2}} < \frac{3}{2}$$

for all integers $n \geq 0$.

SOLUTIONS

A Fibonacci Sine

B-935 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 40, no. 2, May 2002)***

Prove that

$$8\sin\left(\frac{F_3}{2}\right)\sin\left(\frac{F_9}{2}\right)\sin\left(\frac{F_{12}}{2}\right) < 1,$$

where the arguments are measured in degrees.

Solution by Walther Janous, Innsbruck, Austria

In what follows, we shall prove a stronger inequality. We start from the familiar inequality

$$\frac{\sin(x)}{x} < 1$$

valid for all $x \neq 0$.

We observe that, for α measured in degrees, there holds

$$\sin(\alpha) = \sin\left(\frac{\alpha\cdot\pi}{180}\right).$$

Therefore,

$$\sin\left(\frac{F_3}{2}\right)\cdot\sin\left(\frac{F_9}{2}\right)\cdot\sin\left(\frac{F_{12}}{2}\right) < \frac{F_3}{2}\cdot\frac{F_9}{2}\cdot\frac{F_{12}}{2}\cdot\left(\frac{\pi}{180}\right)^3,$$

that is

$$\sin\left(\frac{F_3}{2}\right)\cdot\sin\left(\frac{F_9}{2}\right)\cdot\sin\left(\frac{F_{12}}{2}\right) < \frac{17\pi^3}{81000},$$

whence, finally,

$$8\cdot\sin\left(\frac{F_3}{2}\right)\cdot\sin\left(\frac{F_9}{2}\right)\cdot\sin\left(\frac{F_{12}}{2}\right)<\frac{17\pi^3}{10125}=0.05205992133$$

and the proof is complete.

Several solvers proved that $8\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\le 1$, *where* $x+y+z=\pi$ *with equality occurring when* $x=y=z=\frac{\pi}{3}$.

Also solved by Paul Bruckman, Scott Brown, Haci Civciv, José Luis Diáz-Barrero & Juan José Egozcue (jointly), M. Deshpande, L. A. G. Dresel, Douglas Iannucci, John Jaroma, H.-J. Seiffert, and the proposer.

Exclusive Roots

B-936 ***Proposed by José Luis Diáz & Juan José Egozcue, Terrassa, Spain (Vol. 40, no. 2, May 2002)***

Let n be a nonnegative integer. Show that the equation

$$x^5+F_{2n}x^4+2(F_{2n}-2F_{n+1}^2)x^3+2F_{2n}(F_{2n}-2F_{n+1}^2)x^2+F_{2n}^2x+F_{2n}^3=0$$

has only integer roots.

Solution by Maitland A. Rose, University of South Carolina, Sumter, SC

Use is made of the identities $F_{2n}=F_nL_n$ and $F_n+L_n=2F_{n+1}$, which leads to

$$F_n^2+L_n^2=-2(F_{2n}-2F_{n+1}^2).$$

We note that

$$\begin{aligned}&(x-F_n)(x+F_n)(x-L_n)(x+L_n)(x+F_{2n})\\&=(x^2-F_n^2)(x^2-L_n^2)(x+F_{2n})\\&=(x^4-x^2(F_n^2+L_n^2)+F_n^2L_n^2)(x+F_{2n})\\&=x^5+F_{2n}x^4-(F_n^2+L_n^2)x^3-F_{2n}(F_n^2+L_n^2)x^2+F_n^2L_n^2x+F_{2n}F_n^2L_n^2\\&=x^5+F_{2n}x^4+2(F_{2n}-2F_{n+1}^2)x^3+2F_{2n}(F_{2n}-2F_{n+1}^2)x^2+F_{2n}^2x+F_{2n}^3.\end{aligned}$$

The roots of the given equation are the integers $\pm F_n$, $\pm L_n$, and $-F_{2n}$.

Pentti Haukkanen and Walther Janous used Mathematica *and* Derive, *respectively, to do the calculations and found the same roots.*

Also solved by Paul Bruckman, Charles Cook, Haci Civciv & Naim Tuglu (jointly), M. N. Deshpande, L. A. G. Dresel, Steve Edwards, Ovidiu Furdui, Pentti Haukkanen, Walther Janous, Harris Kwong, Don Redmond, Jaroslav Seibert, H.-J. Seiffert, James Sellers, and the proposers.

Some Identities

B-937 ***Proposed by Paul Bruckman, Sacramento, CA (Vol. 40, no. 2, May 2002)***

Prove the following identities:

(a) $(F_n)^2+(F_{n+1})^2+4(F_{n+2})^2=(F_{n+3})^2+(L_{n+1})^2$;

(b) $(L_n)^2+(L_{n+1})^2+4(L_{n+2})^2=(L_{n+3})^2+(5F_{n+1})^2$.

Solution by Jaroslav Seibert, University of Hradec Králové, The Czech Republic

We will prove the more general identity,

$$G_n^2+G_{n+1}^2+4G_{n+2}^2=G_{n+3}^2+(G_n+G_{n+2})^2,$$

where $\{G_n\}_{n=1}^{\infty}$ is an arbitrary sequence satisfying the recurrence $G_{n+2}=G_{n+1}+G_n$.

The more general identity may be written as

$$\begin{aligned}
&G_n^2+G_{n+1}^2+4G_{n+2}^2-G_{n+3}^2-(G_n+G_{n+2})^2\\
&=G_n^2+G_{n+1}^2+4(G_n+G_{n+1})^2-(G_n+2G_{n+1})^2-(2G_n+G_{n+1})^2\\
&=G_n^2+G_{n+1}^2+4G_n^2+8G_nG_{n+1}+4G_{n+1}^2-G_n^2-4G_nG_{n+1}-4G_{n+1}^2-4G_n^2-4G_nG_{n+1}-G_{n+1}^2=0.
\end{aligned}$$

If we put $G_n=F_n$, then $F_n+F_{n+2}=L_{n+1}$ and we obtain (a).

If we put $G_n=L_n$, then $L_n+L_{n+2}=5F_{n+1}$ and we obtain (b).

Also solved by Scott Brown, Mario Catalani, Haci Civciv, Charles Cook, Kenneth Davenport, M. N. Deshpande, José Luis Diáz-Barrero & Juan José Egozcue (jointly), L. A. G. Dresel, Steve Edwards, Ovidiu Furdui (two solutions), Pentti Haukkanen, Walther Janous, Muneer Jebreel, Harris Kwong, William Moser, Maitland Rose, H.-J. Seiffert, James Sellers, and the proposer.

Series Problem

B-938 *Proposed by Charles K. Cook, University of South Carolina at Sumpter, Sumpter, SC (Vol. 40, no. 2, May 2002)*

Find the smallest positive integer k for which the given series converges and find its sum

(a) $\sum_{n=1}^{\infty}\frac{nF_n}{k^n}$;

(b) $\sum_{n=1}^{\infty}\frac{nL_n}{k^n}$.

Solution by Don Redmond, Southern Illinois University at Carbondale, Carbondale, IL

Let G_n denote either F_n or L_n. If α, β represent the solutions to the quadratic $x^2-x-1=0$, as usual, then for appropriate values of c and d we have $G_n=c\alpha^n+d\beta^n$. Then, if the series converges, we have

$$\begin{aligned}
\sum_{n=1}^{+\infty}\frac{nG_n}{k^n}&=\sum_{n=1}^{+\infty}\frac{n(c\alpha^n+d\beta^n)}{k^n}=c\sum_{n=1}^{+\infty}\frac{n\alpha^n}{k^n}+d\sum_{n=1}^{+\infty}\frac{n\beta^n}{k^n}\\
&=c\frac{(\alpha/k)}{(1-\alpha/k)^2}+d\frac{(\beta/k)}{(1-\beta/k)^2}=c\frac{k\alpha}{(k-\alpha)^2}+d\frac{k\beta}{(k-\beta)^2}.
\end{aligned}$$

Since these are geometric series, we see that they converge if $\max(|\alpha/k|,|\beta/k|)<1$, which gives that the least integer k that yields convergence is $k=2$.

From the definition of α and β, we see that $2-\alpha=\beta^2$ and $2-\beta=\alpha^2$. Thus,

$$\sum_{n=1}^{+\infty}\frac{nG_n}{2^n}=c\frac{2\alpha}{\beta^4}+d\frac{2\beta}{\alpha^4}=2\frac{c\alpha^5+d\beta^5}{(\alpha\beta)^4}=2G_5.$$

Thus, the answer to (a) is $2F_5=10$ and the answer to (b) is $2L_5=22$. □

Problem B-670 also considers these sums.

Also solved by Paul Bruckman , Mario Catalani, Haci Civciv & Naim Tuglu (jointly), Kenneth Davenport, M. N. Deshpande, José Luis Diáz-Barrero & Juan José Egozcue (jointly), L. A. G. Dresel, Steve Edwards, Ovidiu Furdui, Douglas Iannucci, Walther Janous, John Jaroma, Harris Kwong, Kathleen Lewis, Jaroslav Seibert, H.-J. Seiffert, James Sellers, and the proposer.

Identities Problem

B-939 *Proposed by N. Gauthier, Royal Military College of Canada (Vol. 40, no. 2, May 2002)*

For $n \geq 0$ and s arbitrary integers, with

$$f(l,m;n) \equiv f(l,m) = (-1)^{n-l}\binom{n}{l}\binom{n}{m},$$

prove the following identities:

(a) $2^n F_{n+s} = \sum_{l=0}^{4n}\sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m,m)F_{l+s}$;

(b) $3\cdot 2^{n-1}nF_{n+s+2} = \sum_{l=0}^{4n}\sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m,m)[(l-2m)F_{l+s}+mF_{l+s-1}]$.

Solution by H.-J. Seiffert, Berlin, Germany

For $(x,y) \in R^2$, let

$$\begin{aligned} S_n(x,y) &= \sum_{l=0}^{4n}\sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m,m)x^l y^m \\ &= \sum_{l=0}^{4n}\sum_{m=0}^{\lfloor l/3 \rfloor}(-1)^{n-1+m}\binom{n}{l-3m}\binom{n}{m}x^l y^m. \end{aligned}$$

Changing the summations and reindexing gives

$$\begin{aligned} S_n(x,y) &= \sum_{m=0}^{n}\sum_{l=3m}^{n+3m}(-1)^{n-1+m}\binom{n}{l-3m}\binom{n}{m}x^l y^m \\ &= \sum_{m=0}^{n}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{n}{m}x^{k+3m}y^m, \end{aligned}$$

which turns out to be the product of two sums. By the binomial theorem,

$$S_n(x,y) = (1+x^3y)^n(x-1)^n. \tag{1}$$

Proof of (a): From the known Binet form of the Fibonacci numbers, we see that the right-hand side of the desired identity equals

$$A_n = \frac{1}{\sqrt{5}}(S_n(\alpha,1)\alpha^s - S_n(\beta,1)\beta^s).$$

Since $1+\alpha^3 = 2\alpha^2$, $1+\beta^3 = 2\beta^2$, $\alpha-1=-\beta$, $\beta-1=-\alpha$, and $-\alpha\beta=1$, by (1), we have

$$A_n = 2^n F_{n+s}.$$

Proof of (b): Let

$$B_n := \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m, m)(l-3m)F_{l+s}$$

and

$$C_n := \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m, m) m F_{l+s+1}.$$

Consider the function

$$T_n(x, y) := S_n(x, x^{-3}y) = (1+y)^n (x-1)^n.$$

If

$$U_n(x, y) := \frac{\partial T_n(x, y)}{\partial x} = n(1+y)^n (x-1)^{n-1} \tag{2}$$

and

$$V_n(x, y) := \frac{\partial T_n(x, y)}{\partial y} = n(1+y)^{n-1}(x-1)^n \tag{3}$$

denote the partial derivatives of T_n, then, by the definition of S_n,

$$B_n = \frac{1}{\sqrt{5}}(U_n(\alpha, \alpha^3)\alpha^{s+1} - U_n(\beta, \beta^3)\beta^{s+1}),$$

or, by (2), $B_n = 2^n n F_{n+s+2}$. Similarly, from the definition of S_n and (3), one finds

$$C_n = \frac{1}{\sqrt{5}}(V_n(\alpha, \alpha^3)\alpha^{s+4} - V_n(\beta, \beta^3)\beta^{s+4}),$$

or $C_n = 2^{n-1} n F_{n+s+2}$. It follows that $B_n + C_n = 3 \cdot 2^{n-1} n F_{n+s+2}$, which proves the requested identity, because

$$(l-3m)F_{l+s} + mF_{l+s+1} = (l-2m)F_{l+s} + mF_{l+s-1}.$$

Also solved by Paul Bruckman and the proposer.

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