AN INVOLUTORY MATRIX OF EIGENVECTORS

David Callan

Department of Statistics, University of Wisconsin-Madison
1210 W. Dayton Street, Madison, WI 53706-1693

Helmut Prodinger

The John Knopfmacher Centre for Applicable Analysis and Number Theory,
School of Mathematics, University of the Witwatersrand
P.O. Wits, 2050 Johannesburg, South Africa

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1. INTRODUCTION

A matrix with a full set of linearly independent eigenvectors is diagonalizable: if the n by n matrix A has eigenvalues λ_j with corresponding eigenvectors $u_j (1 \leq j \leq n)$, if $U = (u_1|u_2|\ldots|u_n)$ and $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then U is a diagonalizing matrix for A: $U^{-1}AU = D$. Taking transposes shows that $(U^{-1})^t$ is a diagonalizing matrix for A^t. Hence U^t itself is a diagonalizing matrix for A^t if U^2 is the identity matirx, or more generally, due to the scalability of eigenvectors, if U^2 is a scalar matrix.

The purpose of this note is to point out that the right-justified Pascal-triangle matrix $R = \left(\binom{i-1}{n-j}\right)_{1\leq i,j\leq n}$ is an example of this phenomenon. Let a denote the golden ratio $(1+\sqrt{5})/2$.

The eigenvalues of R^t (which of course are the same as the eigenvalues of R) were found in [1]: $\lambda_i = (-1)^{n-i}a^{2i-n-1}$, $1 \leq i \leq n$. The corresponding eigenvectors u_i of R^t were also found in [1] (here suitably scaled for our purposes): $u_i = (u_{ij})_{1\leq j\leq n}$ where

$$u_{ij} = (-a)^{n-j}\sum_{k=1}^{j}(-1)^{i-k}\binom{i-1}{k-1}\binom{n-i}{j-k}a^{2k-i-1}.$$

Let $U = (u_{ij})_{1\leq i,j\leq n}$.

For example, when $n = 5$,

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} a^4 & -4a^3 & 6a^2 & -4a & 1 \\ -a^3 & 3a^2-a^4 & -3a+3a^3 & 1-3a^2 & a \\ a^2 & -2a+2a^3 & 1-4a^2+a^4 & 2a-2a^3 & a^2 \\ -a & 1-3a^2 & 3a-3a^3 & 3a^2-a^4 & a^3 \\ 1 & 4a & 6a^2 & 4a^3 & a^4 \end{pmatrix}.$$

Since the rows of U are eigenvectors of R^t, U^t is a diagonalizing matrix for R^t. By the first paragraph applied to $A = R^t$, U will be a diagonalizing matrix for R if we can show that

$(U^t)^2$ (equivalently U^2) is a scalar matrix. We now proceed to show that $U^2 = (1+a^2)^{n-1} I_n$ and in fact this holds for arbitrary a. We use the notation $[x^k]p(x)$ to denote the coefficient of x^k in the polynomial $p(x)$. Consider the generating function $U_i(z) = z(z-a)^{n-i}(az+1)^{i-1}$. Using the binomial theorem to expand $U_i(z)$, it is immediate that

$$U_i(z) = \sum_{j=1}^{n} u_{ij} z^j.$$

Now the (i,k) entry of U^2 is

$$(U^2)_{ik} = \sum_{j=1}^{n} \left[x^{k-1}\right](x-a)^{n-j}(ax+1)^{j-1} \cdot \left[z^{j-1}\right](z-a)^{n-i}(az+1)^{i-1}$$

$$= \left[x^{k-1}\right] \sum_{j=1}^{n} \left[z^{j-1}\right](x-a)^{n-j}(ax+1)^{j-1}(z-a)^{n-i}(az+1)^{i-1}$$

$$= \left[x^{k-1}\right](x-a)^{n-1} \sum_{j=1}^{n} \left[z^{j-1}\right] \left(\frac{ax+1}{x-a}\right)^{j-1} (z-a)^{n-i}(az+1)^{i-1}$$

$$= \left[x^{k-1}\right](x-a)^{n-1} \sum_{j=1}^{n} \left[z^{j-1}\right] \left(z\frac{ax+1}{x-a} - a\right)^{n-i} \left(az\frac{ax+1}{x-a} + 1\right)^{i-1}$$

$$= \left[x^{k-1}\right](x-a)^{n-1} \left(\frac{ax+1}{x-a} - a\right)^{n-i} \left(a\frac{ax+1}{x-a} + 1\right)^{i-1}$$

$$\begin{aligned} &= \left[x^{k-1}\right](ax+1-ax+a^2)^{n-i}(a^2x+a+x-a)^{i-1} \\ &= \left[x^{k-1}\right](1+a^2)^{n-i}x^{i-1}(1+a^2)^{i-1} \\ &= \left[x^{k-i}\right](1+a^2)^{n-1} \\ &= (1+a^2)^{n-1}\delta_{ki}, \end{aligned}$$

as desired.

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REFERENCES

[1] R.S. Melham and C. Cooper. "The Eigenvectors of a Certain Matrix of Binomial Coefficients." *The Fibonacci Quarterly* **38.2** (2000):123-26.

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