ON RATIONAL APPROXIMATIONS BY PYTHAGOREAN NUMBERS

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1. STATEMENT OF THE RESULTS

A famous result of *Heilbronn* states that for every real irrational ξ and any $\varepsilon > 0$ there are infinitely many integers n satisfying

$$\| \xi n^2 \| < \frac{1}{n^{1/2-\varepsilon}}.$$

Here $\| \cdot \|$ denotes the distance to the nearest integer [3]. In view of our results below we reformulate Heilbronn's theorem as follows: There are infinitely many pairs of integers m, k where m is a perfect square such that the inequality

$$|\xi m - k| < \frac{1}{m^{1/4-\varepsilon}}$$

holds.

The Pythagorean numbers x, y, z with $x^2 + y^2 = z^2$, where additionally x and y are coprime, play an important role in number theory since they were first investigated by the ancients. It is well-known that to every Pythagorean triplet x, y, z of positive integers satisfying

$$x^2 + y^2 = z^2, \quad (x, y) = 1, \quad x \equiv 0 \bmod 2 \tag{1.1}$$

a pair of positive integers a, b with $a > b > 0$ corresponds such that

$$x = 2ab, \; y = a^2 - b^2, \; z = a^2 + b^2, \; (a, b) = 1, \; a + b \equiv 1 \bmod 2 \tag{1.2}$$

hold ([2], Theorem 225). Moreover, there is a (1,1) correspondence between different values of a, b and different values of x, y, z. The object of this paper is to investigate diophantine inequalities $|\xi y - x|$ for integers y and x from triplets of Pythagorean numbers. Since $x^2 + y^2$ is required to be a perfect square - in what follows we write $x^2 + y^2 \in \square$ - we have a essential restriction on the rationals x/y approximating a real irrational ξ. So one may not expect to get a result as strong as Heilbronn's theorem. Indeed, there are irrationals ξ such that $|\xi y - x| \gg 1$ holds for all integers x, y satisfying $x^2 + y^2 \in \square$. But almost all real irrationals ξ (in the sense of the Lebesgue-measure) can be approximated in such a way that $|\xi y - x|$ tends to zero for a infinite sequence or pairs x, y corresponding to Pythagorean numbers. In order to prove our results we shall make use of the properties of continued fraction expansions. By our first theorem we describe those real irrationals having good approximations by Pythagorean numbers.

Theorem 1.1: *Let $\xi > 0$ denote a real irrational number such that the quotients of the continued fraction expansion of at least one of the numbers $\eta_1 := \xi + \sqrt{1+\xi^2}$ and $\eta_2 := (1+\sqrt{1+\xi^2})/\xi$ are not bounded. Then there are infinitely many pairs of positive integers x, y satisfying*

$$|\xi y - x| = o(1) \quad \text{and} \quad x^2 + y^2 \in \square\,.$$

Conversely, if the quotients of both of the numbers η_1 and η_2 are bounded, then there exists some $\delta > 0$ such that

$$|\xi y - x| \geq \delta$$

holds for all positive integers x, y where $x^2 + y^2 \in \square$.

It can easily be seen that the irrationality of ξ does not allow the numbers η_1 and η_2 to be rationals. The following result can be derived from the preceding theorem and from the metric theory of continued fractions:

Corollary 1.1: *To almost all real numbers ξ (in the sense of the Lebesgue measure) there are infinitely many pairs of integers $x \neq 0$, $y > 0$ satisfying*

$$|\xi y - x| = o(1) \qquad \text{and } x^2 + y^2 \in \square\,.$$

Many exceptional numbers ξ not belonging to that set of full measure are given by certain quadratic surds:

Corollary 1.2: *Let $r > 1$ denote some rational such that $\xi := \sqrt{r^2-1}$ is an irrational number. Then the inequality*

$$|\xi y - x| > \delta \tag{1.3}$$

holds for some $\delta > 0$ (depending only on r) and for all positive integers x, y with $x^2 + y^2 \in \square$.

The lower bound δ can be computed explicitly. The corollary follows from Theorem 1.1 by setting $\xi := \sqrt{r^2-1}$.

Taking $r = 3/2$, we conclude that $\xi = \sqrt{5}/2$ satisfies the condition of Corollary 1.2. Involving some refinements of the estimates from the proof of the general theorem, we find that (1.3) holds with $\delta = 1/4$ for $\xi = \sqrt{5}/2$.

Finally, we give an application to *inhomogeneous diophantine approximations* by Fibonacci numbers. Although $|y\sqrt{5}/2 - x| > \delta$ holds for all Pythagorean numbers x, y, this is no longer true in the case of inhomogeneous approximation. By the following result we estimate $|\xi y - x - \eta|$ for infinitely many Pythagorean numbers x and y, where ξ and η are given by $F_k\sqrt{5}/2$ and $\pm F_{2k}/\sqrt{5}$, respectively, for some fixed even integer k.

Theorem 1.2: *Let $k \geq 2$ denote an even integer. Then,*

$$0 < \frac{F_k\sqrt{5}}{2} \cdot (2F_nF_{n+k}) - F_kF_{2n+k} + (-1)^n\frac{F_{2k}}{\sqrt{5}} < \frac{2^{2n+1}}{5(1+\sqrt{5})^{2n}}$$

holds for all integers $n \geq 1$, and we have

$$(2F_nF_{n+k})^2 + (F_kF_{2n+k})^2 \in \square \ \ (n \geq 1).$$

2. PROOF OF THEOREM 1.1

It can easily be verified that $\eta_1 > 1$ and $\eta_2 > 1$. One gets η_2 by substituting $1/\xi$ for ξ in η_1. First we assume that the sequence $a_0, a_1, a_2, \ldots$ of quotients from the continued fraction expansion $\eta_1 = \langle a_0; a_1, a_2, \ldots \rangle$ is not bounded. If p_n/q_n denotes the n^{th} convergent of η_1, the inequality

$$\left|\eta_1 - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2} \quad (n \geq n_0) \tag{2.1}$$

holds, where n_0 is chosen sufficiently large. There exists some positive real number β such that $\eta_1 = 1 + 2\beta$; particularly we have $\eta_1 > (1+\beta)(1+1/p_n)$ for $n \geq n_0$. By $\eta_1 q_n - p_n < 1$, one gets

$$q_n < \frac{p_n + 1}{\eta_1} < \frac{p_n}{1+\beta} \quad (n \geq n_0). \tag{2.2}$$

Let

$$f(t) := \xi - \frac{1}{2}\left(t - \frac{1}{t}\right) \quad (t \geq 1).$$

By straightforward computations it can easily be verified that

$$f(\eta_1) = 0. \tag{2.3}$$

For any two real numbers t_1, t_2 satisfying $1 \leq t_1 < t_2$ there is some real number α with $t_1 \leq \alpha \leq t_2$ such that

$$|f(t_2) - f(t_1)| = |f'(\alpha)| \cdot |t_2 - t_1|$$

holds. In the case when n is even let $t_1 = p_n/q_n$ and $t_2 = \eta_1$, otherwise put $t_1 = \eta_1$, $t_2 = p_n/q_n$. Thus, for any even index $n \geq n_0$ we have $\eta_1 \geq \alpha \geq p_n/q_n > 1$, where the lower bound 1 follows immediately from $\eta_1 > 1$ and from (2.1). For any odd index n it is clear that $\alpha \geq \eta_1$ holds. Therefore one gets

$$\left|f(\eta_1) - f\left(\frac{p_n}{q_n}\right)\right| = \frac{1}{2}\left(1 + \frac{1}{\alpha^2}\right) \cdot \left|\eta_1 - \frac{p_n}{q_n}\right|$$

where

$$\alpha > 1 \quad (n \geq n_0). \tag{2.4}$$

Applying (2.1), (2.3), and the definition of f, the inequality takes the form

$$\left|\xi - \frac{1}{2}\left(\frac{p_n}{q_n} - \frac{q_n}{p_n}\right)\right| < \left(1 + \frac{1}{\alpha^2}\right)\frac{1}{2a_{n+1}q_n^2}.$$

Put $x := p_n^2 - q_n^2$, $y := 2p_nq_n$. For n tending to infinity the positive integers p_n are not bounded, therefore we get infinitely many pairs x, y of positive integers. By (2.2), $x > 0$ holds for all sufficiently large indices n. Putting x and y into the above inequality and applying (2.4), we get

$$\left|\xi - \frac{x}{y}\right| < \frac{1}{a_{n+1}q_n^2} = \frac{4p_n^2}{a_{n+1}y^2}, \tag{2.5}$$

where $x^2 + y^2 = (p_n^2 + q_n^2)^2 \in \square$. Using (2.2), we compute an upper bound for p_n^2 on the right side of (2.5): $p_n^2 = x + q_n^2 < x + p_n^2/(1+\beta)^2$, or, $p_n^2 < (1+\beta)^2 x/\beta(2+\beta) < (1+\beta)^2 x/2\beta$ for $n \geq n_0$. Moreover, (2.5) gives $|\xi y - x| < y$, from which the estimate $x \leq (1+\xi)y$ follows immediately. Altogether we have proved that infinitely many pairs of positive integers x, y with $x^2 + y^2 \in \square$ exists such that

$$\left|\xi - \frac{x}{y}\right| < \frac{2(1+\xi)(1+\beta)^2}{\beta a_{n+1} y}$$

holds, where any pair x, y corresponds to some n. Finally, we restrict n on integers from a subsequence corresponding to monotonously increasing partial quotients a_{n+1}. For n tending to infinity, the assertion of the first part of the theorem concerning η_1 follows from

$$|\xi y - x| < \frac{2(1+\xi)(1+\beta)^2}{\beta a_{n+1}}.$$

Next, if the sequence of quotients from the continued fraction expansion of η_2 is not bounded, we get by the same method infinitely many pairs x, y of integers (where y is even) satisfying $x^2 + y^2 \in \square$ and

$$\left|\frac{y}{\xi} - x\right| < \frac{2(1+1/\xi)(1+\beta)^2}{\beta a_{n+1}} \quad (2\beta := \eta_2 - 1,\ \eta_2 = \langle a_0; a_1, a_2, \ldots\rangle).$$

This inequality can be simplified by

$$|\xi x - y| < \frac{2(1+\xi)(1+\beta)^2}{\beta a_{n+1}},$$

which completes the proof of the first part of the theorem.

In order to show the second part we now assume that both numbers, η_1 and η_2, have bounded partial quotients. It suffices to prove

$$|\xi y - x| > \delta \tag{2.6}$$

for coprime Pythagorean integers x and y : if $|\xi y - x| \leq \delta$ for $(x,y) > 1$, one may divide the inequality by (x,y). Then we get a new pair of coprime integers with

$$\left(\frac{x}{(x,y)}\right)^2 + \left(\frac{y}{(x,y)}\right)^2 \in \square,$$

which contradicts (2.6). From the hypothesis on η_1 we conclude that there is some positive real number δ_1 satisfying

$$\left|\eta_1 - \frac{a}{b}\right| > \frac{\delta_1}{b^2} \tag{2.7}$$

for all positive coprime integers a and b.

The first assertion we shall disprove states that there are infinitely many pairs of positive coprime integers x, y such that $2|y,\ x^2 + y^2 \in \square$, and

$$|\xi y - x| < \delta_1(\eta_1 - 1). \tag{2.8}$$

By (1.1) and (1.2) we know that to every pair x, y two integers a, b correspond such that $y = 2ab,\ x = a^2 - b^2,\ a > b,\ (a,b) = 1$, and $a + b \equiv 1 \bmod 2$. Again we denote by $f(t)(t \geq 1)$ the function defined above. Using $\eta_1 > 1$ and $a/b > 1$ it is clear that $f'(t)$ is defined for all real numbers which are situated between η_1 and a/b. Therefore, corresponding to a and b, a real number α exists satisfying

$$(\alpha - \eta_1) \cdot \left(\alpha - \frac{a}{b}\right) < 0 \text{ and } \left|f(\eta_1) - f\left(\frac{a}{b}\right)\right| = |f'(\alpha)| \cdot \left|\eta_1 - \frac{a}{b}\right|.$$

By (2.3) we find that

$$\left|\xi - \frac{x}{y}\right| = \left|\xi - \frac{a^2 - b^2}{2ab}\right| = \frac{1}{2}\left(1 + \frac{1}{\alpha^2}\right) \cdot \left|\eta_1 - \frac{a}{b}\right|. \tag{2.9}$$

In what follows it is necessary to distinguish two cases.

Case 1: $|\eta_1 - a/b| \geq 1$.

Using $1/\alpha > 0$, we conclude from (2.9) that $|\xi y - x| > y/2$. For all sufficiently large integers y this contradicts to our assumption from (2.8).

Case 2: $|\eta_1 - a/b| < 1$.

First, it follows from this hypothesis that $b < a/(\eta_1 - 1)$. Next, we estimate the right side of (2.9) by the inequality from (2.7):

$$\left|\xi - \frac{x}{y}\right| > \frac{\delta_1}{2b^2} > \frac{\delta_1(\eta_1 - 1)}{2ab}.$$

Consequently we have, using $y = 2ab$,

$$|\xi y - x| > \delta_1(\eta_1 - 1),$$

which again is impossible by our assumption. So we have proved that there are at most finitely many pairs x, y of positive coprime integers satisfying $2|y,\ x^2 + y^2 \in \square$, and

$$|\xi y - x| < \delta_1(\eta_1 - 1). \tag{2.10}$$

Since we may assume that the partial quotients of the number η_2 are also bounded, we get a similar result concerning the approximation of $1/\xi$: There are at most finitely many pairs x, y of positive coprime integers with $2|y,\ x^2 + y^2 \in \square$, and

$$\left|\frac{y}{\xi} - x\right| < \delta_2(\eta_2 - 1), \tag{2.11}$$

where δ_2 denotes some positive real number satisfying

$$\left|\eta_2 - \frac{a}{b}\right| > \frac{\delta_2}{b^2}$$

for all coprime positive integers a and b. Since ξ is positive, the inequality from (2.11) can be transformed into

$$|\xi x - y| < \delta_2 \xi(\eta_2 - 1),$$

which is satisfied at most by finitely many coprime Pythagorean numbers x, y with $2|y$. By (2.10) we complete the proof of the theorem.

3. PROOF OF THEOREM 1.2

Lemma 3.1: *Let $k \geq 2$ and $n \geq 1$ denote integers, where k is even. Then one has*

$$F_{n+k}^2 - F_n^2 = F_k F_{2n+k} \tag{3.1}$$

and

$$(2F_n F_{n+k})^2 + (F_k F_{2n+k})^2 \in \square. \tag{3.2}$$

Proof: Throughout this final section we denote the number $(1 + \sqrt{5})/2$ by ρ. We shall need *Binet's formula*

$$F_m = \frac{1}{\sqrt{5}}\left(\rho^m - \frac{(-1)^m}{\rho^m}\right) \quad (m \geq 1). \tag{3.3}$$

Since k is assumed to be even, one gets from (3.3):

$$5(F_{n+k}^2 - F_n^2) = \left(\rho^{n+k} - \frac{(-1)^n}{\rho^{n+k}}\right)^2 - \left(\rho^n - \frac{(-1)^n}{\rho^n}\right)^2$$

$$= \left(\rho^k - \frac{1}{\rho^k}\right) \cdot \left(\rho^{2n+k} - \frac{1}{\rho^{2n+k}}\right) = 5F_k F_{2n+k}.$$

This proves (3.1). Then the second assertion of the lemma follows easily, since one has

$$(2F_nF_{n+k})^2 + (F_kF_{2n+k})^2 = (2F_nF_{n+k})^2 + (F_{n+k}^2 - F_n^2)^2 = (F_{n+k}^2 + F_n^2)^2 \in \square.$$

Binet's formula (3.3) is a basic identity which also is used a several times to prove the inequalities in Theorem 1.2. Since $k \geq 2$ is assumed to be an even integer, one gets

$$\sqrt{5}F_kF_nF_{n+k} - F_kF_{2n+k} + (-1)^n\frac{F_{2k}}{\sqrt{5}}$$

$$= \frac{1}{5}\left(\rho^k - \frac{1}{\rho^k}\right) \cdot \left(\rho^n - \frac{(-1)^n}{\rho^n}\right) \cdot \left(\rho^{n+k} - \frac{(-1)^n}{\rho^{n+k}}\right) -$$

$$-\frac{1}{5}\left(\rho^k - \frac{1}{\rho^k}\right) \cdot \left(\rho^{2n+k} - \frac{1}{\rho^{2n+k}}\right) + \frac{(-1)^n}{5}\left(\rho^{2k} - \frac{1}{\rho^{2k}}\right)$$

$$= \frac{2}{5}\left(1 - \frac{1}{\rho^{2k}}\right)\frac{1}{\rho^{2n}}.$$

It follows that the term on the left side represents a positive real number, which is bounded by $2/5\rho^{2n}$. By (3.2) from Lemma 3.1, this finishes the proof of the theorem.

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