GENERALIZED FIBONACCI FUNCTIONS AND SEQUENCES OF GENERALIZED FIBONACCI FUNCTIONS

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1. INTRODUCTION

We consider a generalization of the Fibonacci sequence which is called the k-Fibonacci sequence for a positive integer $k \geq 2$. The *k-Fibonacci sequence* $\{g_n^{(k)}\}$ is defined as

$$g_0^{(k)} = g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0, \quad g_{k-1}^{(k)} = 1$$

and for $n \geq k \geq 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.$$

We call $g_n^{(k)}$ the n^{th} *k-Fibonacci number*. For example, if $k = 2$, then $\{g_n^{(2)}\}$ is the Fibonacci sequence $\{F_n\}$. If $k = 5$, then $g_0^{(5)} = g_1^{(5)} = g_2^{(5)} = g_3^{(5)} = 0$, $g_4^{(5)} = 1$, and the 5-Fibonacci sequence is

$$\left(g_0^{(5)} = 0\right), 0, 0, 0, 1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, \ldots.$$

Let E be a 1 by $(k-1)$ matrix whose entries are ones and let I_n be the identity matrix of order n. Let $\mathbf{g}_n^{(k)} = \left(g_n^{(k)}, \ldots, g_{n+k-1}^{(k)}\right)^T$ for $n \geq 0$. For any $k \geq 2$, the fundamental recurrence relation, $n \geq k$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}$$

can be defined by the vector recurrence relation $\mathbf{g}_{n+1}^{(k)} = Q_k \mathbf{g}_n^{(k)}$, where

$$Q_k = \begin{bmatrix} 0 & I_{k-1} \\ 1 & E \end{bmatrix}. \tag{1}$$

We call Q_k the *k-Fibonacci matrix*. By applying (1), we have $\mathbf{g}_{n+1}^{(k)} = Q_k^n \mathbf{g}_1^{(k)}$. In [4], [6] and [7], we can find relationships between the k-Fibonacci numbers and their associated matrices.

In [2], M. Elmore introduced the Fibonacci function following as:

$$f_0(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\sqrt{5}}, \; f_n(x) = f_0^{(n)}(x) = \frac{\lambda_1^n e^{\lambda_1 x} - \lambda_2^n e^{\lambda_2 x}}{\sqrt{5}},$$

and hence $f_{n+1}(x) = f_n(x) + f_{n-1}(x)$, where

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

Here, λ_1, λ_2 are the roots of $x^2 - x - 1 = 0$.

In this paper, we consider a function which is a generalization of the Fibonacci function and consider sequences of generalized Fibonacci functions.

2. GENERALIZED FIBONACCI FUNCTIONS

For positive integers l and n with $l \leq n$, let $Q_{l,n}$ denote the set of all strictly increasing l-sequences from $\{1, 2, \ldots, n\}$. For an $n \times n$ matrix A and for $\alpha, \beta \in Q_{l,n}$, let $A[\alpha|\beta]$ denote the matrix lying in rows α and columns β and let $A(\alpha|\beta)$ denote the matrix complementary to $A[\alpha|\beta]$ in A. In particular, we denote $A(\{i\}|\{j\}) = A(i|j)$.

We define a function $G(k, x)$ by

$$G(k,x) = \sum_{i=0}^{\infty} \frac{g_i^{(k)}}{i!} x^i.$$

Since

$$\lim_{n\to\infty} \frac{g_n^{(k)}(n+1)}{g_{n+1}^{(k)}} \to \infty,$$

the function $G(k, x)$ is convergent for all real number x.

For fixed $k \geq 2$, the power series $G(k, x)$ satisfies the differential equation

$$G^{(k)}(k,x) - G^{(k-1)}(k,x) - \cdots - G''(k,x) - G'(k,x) - G(k,x) = 0. \qquad (2)$$

In [5], we can find that the characteristic equation $x^k - x^{k-1} - \cdots - x - 1 = 0$ of Q_k does not have multiple roots. So, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of $x^k - x^{k-1} - \cdots - x - 1 = 0$, then

$\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. That is, the eigenvalues of Q_k are distinct. Define V to be the k by k Vandermonde matrix by

$$V = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{k-2} & \lambda_2^{k-2} & \ldots & \lambda_k^{k-2} \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \ldots & \lambda_k^{k-1} \end{bmatrix}. \tag{2}$$

Then we have the following theorem.

Theorem 2.1: Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of the k-Fibonacci matrix Q_k. Then, the initial-value problem $\sum_{i=0}^{k-1} G^{(i)}(k,x) = G^{(k)}(k,x)$, where $G^{(i)}(k,0) = 0$ for $i = 0, 1, \ldots, k-2$, and $G^{(k-1)}(k,0) = 1$ has the unique solution $G(k,x) = \sum_{i=1}^{k} c_i e^{\lambda_i x}$, where

$$c_i = (-1)^{k+i} \frac{\det V(k|i)}{\det V}, \; i = 1, 2, \ldots, k. \tag{3}$$

Proof: Since the characteristic equation of Q_k is $x^k - x^{k-1} - \cdots - x - 1 = 0$, it is clear that $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_k e^{\lambda_k x}$ is a solution of (2.).

Now, we will prove that $c_i = \frac{1}{\det V}(-1)^{k+i} \det V(k|i), i = 1, 2, \ldots, k$. Since $G(k,x) = c_1 e^{\lambda_i x} + c_2 e^{\lambda_2 x} + \ldots c_k e^{\lambda_k x}$ and for $x = 0$, $G^{(i)}(k,0) = 0$ for $i = 0, 1, \ldots, k-2, G^{(k-1)}(k,0) = 1$, we have

$$G(k,0) = c_1 + c_2 + \cdots + c_k = 0$$
$$G'(k,0) = c_1\lambda_1 + c_2\lambda_2 + \cdots + c_k\lambda_k = 0$$
$$\vdots$$
$$G^{(k-2)}(k,0) = c_1\lambda_1^{k-2} + c_2\lambda_2^{k-2} + \cdots + c_k\lambda_k^{k-2} = 0$$
$$G^{(k-1)}(k,0) = c_1\lambda_1^{k-1} + c_2\lambda_2^{k-1} + \cdots + c_k\lambda_k^{k-1} = 1.$$

Let $\mathbf{c} = (c_1, c_2, \ldots, c_{k-1}, c_k)^T$ and $\mathbf{b} = (0, 0, \ldots, 0, 1)^T$. Then we have $V\mathbf{c} = \mathbf{b}$. Since the matrix V is a Vandermonde matrix and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, the matrix V is nonsingular. For $i = 1, 2, \ldots, k$, the matrix $V(k|i)$ is also a Vandermonde matrix and nonsingular. Therefore, by Cramer's rule, we have $c_i = (-1)^{k+i} \frac{\det V(k|i)}{\det V}$, $i = 1, 2, \ldots, k$ and the proof is complete. □

We can replace the writing of (2) by the form

$$G^{(k)}(k,x) = G^{(k-1)}(k,x) + \cdots + G''(k,x) + G'(k,x) + G(k,x).$$

This suggests that we use the notation $G_0(k,x) = G(k,x)$ and, for $i \geq 1, G_i(k,x) = G^{(i)}(k,x)$. Thus

$$G_n(k,x) = G^{(n)}(k,x) = c_1\lambda_1^n e^{\lambda_1 x} + c_2\lambda_2^n e^{\lambda_2 x} + \cdots + c_k\lambda_k^n e^{\lambda_k x}$$

gives us the sequence of functions $\{G_n(k,x)\}$ with the property that

$$G_n(k,x) = G_{n-1}(k,x) + G_{n-2}(k,x) + \cdots + G_{n-k}(k,x), \quad n \geq k, \tag{4}$$

where each c_i is in (3). We shall refer to these functions as *k-Fibonacci functions*. If $k = 2$, then $G(2,x) = f_0(x)$ is the Fibonacci function as in [2]. From (4), we have the following theorem.

Theorem 2.2: For the k-Fibonacci function $G_n(k,x)$,

$$G_0(k,0) = 0 = g_0^{(k)}, G_1(k,0) = 0 = g_1^{(k)}, \ldots, G_{k-2}(k,0) = 0 = g_{k-2}^{(k)},$$

$$G_{k-1}(k,0) = 1 = g_{k-1}^{(k)}, G_k(k,0) = G_0(k,0) + \cdots + G_{k-1}(k,0) = 1 = g_k^{(k)},$$

$$\begin{aligned} g_n^{(k)} &= G_n(k,0) = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_k\lambda_k^n \\ &= g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}, \; n \geq k, \end{aligned}$$

where each c_i is given by (3).

Let $\mathbb{G}_n(k,x) = (G_n(k,x), \ldots, G_{n+k-1}(k,x))^T$. For $k \geq 2$, the fundamental recurrence realtion (4) can be defined by the vector recurrence relation $\mathbb{G}_{n+1}(k,x) = Q_k\mathbb{G}_n(k,x)$ and hence $\mathbb{G}_{n+1}(k,x) = Q_k^n\mathbb{G}_1(k,x)$.

Since $g_{k-1}^{(k)} = g_k^{(k)} = 1$, we can replace the matrix Q_k in (1) with

$$Q_k = \begin{bmatrix} 0 & g_{k-1}^{(k)} & 0 & \cdots & 0 \\ 0 & 0 & g_{k-1}^{(k)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & g_{k-1}^{(k)} \\ g_{k-1}^{(k)} & g_{k-1}^{(k)} & \cdots & g_{k-1}^{(k)} & g_k^{(k)} \end{bmatrix}.$$

Then we can find the matrix $Q_k^n = [g_{i,j}^{\dagger}(n)]$ in [5] where, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, k$,

$$g_{i,j}^{\dagger}(n) = g_{n+(i-2)}^{(k)} + \cdots + g_{n+(i-2)-(j-1)}^{(k)}. \tag{5}$$

We know that $g_{i,1}^{\dagger}(n) = g_{n+i-2}^{(k)}$ and $g_{i,k}^{\dagger}(n) = g_{n+i-1}^{(k)}$. So, we have the following theorem.

Theorem 2.3: For nonnegative integers n and m, $n+m \geq k$, we have

$$G_{n+m+1}(k,x) = \sum_{j=1}^{k} g_{1,j}^{\dagger}(n) G_{m+j}(k,x).$$

In particular,

$$G_k(k,x) = \sum_{i=0}^{\infty} \frac{g_{i+k}^{(k)}}{i!} x^i.$$

Proof: Since $\mathbf{G}_{n+1}(k,x) = Q_k^n \mathbf{G}_1(k,x)$,

$$\begin{aligned} G_{n+m+1}(k,x) &= Q_k^{n+m} \mathbf{G}_1(k,x) = Q_k^n \cdot Q_k^m \mathbf{G}_1(k,x) \\ &= Q_k^n \mathbf{G}_{m+1}(k,x). \end{aligned}$$

By applying (5), we have

$$G_{n+m+1}(k,x) = g_{1,1}^{\dagger}(n) G_{m+1}(k,x) + \cdots + g_{1,k}^{\dagger}(n) G_{m+k}(k,x).$$

Since $\sum_{i=0}^{k-1} G_i(k,x) = G_k(k,x)$ and

$$\sum_{i=0}^{k-1} G_i(k,x) = g_k^{(k)} + g_{k+1}^{(k)} x + \frac{g_{k+2}^{(k)}}{2!} x^2 + \cdots + \frac{g_{n+k}^{(k)}}{n!} x^n + \ldots,$$

we have

$$G_k(k,x) = \sum_{i=0}^{\infty} \frac{g_{i+k}^{(k)}}{i!} x^i. \quad \square$$

Note that $Q_k^{n+m} = Q_k^{m+n}$. Then we have the following corollary.

Corollary 2.4: For nonnegative integers n and m, $n+m \geq k$, we have

$$G_{n+m+1}(k,x) = \sum_{j=1}^{k} g_{1,j}^{\dagger}(m) G_{n+j}(k,x).$$

We know that the characteristic polynomial of Q_k is $\lambda^k - \lambda^{k-1} - \cdots - \lambda - 1$. So, we have the following lemma.

Lemma 2.5: Let $\lambda^k - \lambda^{k-1} - \cdots - \lambda - 1 = 0$ be the characteristic equation of Q_k. Then, for any root λ of the characteristic equation, $n \geq k > 0$, we have,

$$\lambda^n = \sum_{j=1}^{k} g_{1,j}^{\dagger}(n)\lambda^{j-1}.$$

Proof: From (5) we have, for $j = 1, 2, \ldots, k$,

$$g_{1,j}^{\dagger}(n) = g_{n-1}^{k} + g_{n-2}^{k} + \cdots + g_{n-j}^{k}.$$

It can be shown directly for $n = k$ that

$$\lambda^k = g_k^{(k)}\lambda^{k-1} + \left(g_{k-1}^{(k)} + g_{k-2}^{(k)} + \cdots + g_1^{(k)}\right)\lambda^{k-2} + \cdots + \left(g_{k-1}^{(k)} + g_{k-2}^{(k)}\right)\lambda + g_{k-1}^{k}$$

$$= \lambda^{k-1} + \lambda^{k-2} + \cdots + \lambda + 1$$

We show this by induction on n. Then

$$\lambda^{n+1} = \lambda^n \cdot \lambda$$

$$= \left(g_{1,k}^{\dagger}(n)\lambda^{k-1} + g_{1,k-1}^{\dagger}(n)\lambda^{k-2} + \cdots + g_{1,2}^{\dagger}(n)\lambda + g_{1,1}^{\dagger}(n)\right)\lambda$$

$$= g_n^k\lambda^k + \left(g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k+1}^{(k)}\right)\lambda^{k-1}$$

$$+ \left(g_{n-1}^{(k)} + \cdots + g_{n-k+2}^{(k)}\right)\lambda^{k-2} + \cdots + \left(g_{n-1}^{(k)} + (g_{n-2}^{(k)}\right)\lambda^2 + g_{n-1}^{(k)}\lambda.$$

Since $\lambda^k = \lambda^{k-1} + \cdots + \lambda + 1$, we have

$$\lambda^{n+1} = g_n^{(k)}\left(\lambda^{k-1} + \cdots + \lambda + 1\right) + \left(g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k+1}^{(k)}\right)\lambda^{k-1} +$$

$$\left(g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k+2}^{(k)}\right)\lambda^{k-2} + \cdots + \left(g_{n-1}^{(k)} + g_{n-2}^{(k)}\right)\lambda^2 + g_{n-1}^{(k)}\lambda$$

$$= \left(g_n^{(k)} + g_{n-1}^{(k)} + \cdots + g_{n-k+1}^{(k)}\right)\lambda^{k-1} + \left(g_n^{(k)} + \cdots + g_{n-k+2}^{(k)}\right)\lambda^{k-2}$$

$$+ \cdots + \left(g_n^{(k)} + g_{n-1}^{(k)}\right)\lambda + g_n^{(k)}$$

$$= g_{n+1}^{(k)}\lambda^{k-1} + \left(g_n^{(k)} + g_{n-1}^{(k)} + \cdots + g_{n-k+2}^{(k)}\right)\lambda^{k-2}$$

$$+ \cdots + \left(g_n^{(k)} + g_{n-1}^{(k)}\right)\lambda + g_n^{(k)}$$

$$= g_{1,k}^{\dagger}(n+1)\lambda^{k-1} + g_{1,k-1}^{\dagger}(n+1)\lambda^{k-2} + g_{1,k-2}^{\dagger}(n+1)\lambda^{k-3}$$

$$+ \cdots + g_{1,2}^{\dagger}(n+1)\lambda + g_{1,1}^{\dagger}(n+1)$$

$$= \sum_{j=1}^{k} g_{1,j}^{\dagger}(n+1)\lambda^{j-1}.$$

Therefore, by induction of n, the proof is completed. □

Theorem 2.6: Let λ be a root of characteristic equation of Q_k. For positive integer n, we have

$$G_n(k,\lambda) = \sum_{j=n}^{k} \alpha_{nj}\lambda^{j-1},$$

where

$$\alpha_{j,n} = \frac{g_{n+k}^{(k)}}{k!} + \frac{g_{n+j-1}^{(k)}}{(j-1)!} + \sum_{i=k+1}^{\infty} g_{1,j}^{\dagger}(i)\frac{g_{n+i}^{(k)}}{i!}.$$

Proof: Since $\lambda^k = \lambda^{k-1} + \cdots + \lambda + 1$ and by lemma 2.5, we have

$$\begin{aligned}
G_n(k,\lambda) &= g_n^{(k)} + g_{n+1}^{(k)}\lambda + \frac{g_{n+2}^{(k)}}{2!}\lambda^2 + \cdots + \frac{g_{2n}^{(k)}}{n!}\lambda^n + \dots \\
&= \left(g_n^{(k)} + \frac{g_{n+k}^{(k)}}{k!} + g_{11}^{\dagger}(k+1)\frac{g_{n+k+1}^{(k)}}{(k+1)!} + \cdots + g_{11}^{\dagger}(n)\frac{g_{2n}^{(k)}}{n!} + \dots\right) + \\
&\quad \left(g_{n+1}^{(k)} + \frac{g_{n+k}^{(k)}}{k!} + g_{12}^{\dagger}(k+1)\frac{g_{n+k+1}^{(k)}}{(k+1)!} + \cdots + g_{12}^{\dagger}(n)\frac{g_{2n}^{(k)}}{n!} + \dots\right)\lambda \\
&\quad + \cdots + \\
&\quad \left(\frac{g_{n+k-1}^{(k)}}{(k-1)!} + \frac{g_{n+k}^{(k)}}{k!} + g_{1k}^{\dagger}(k+1)\frac{g_{n+k+1}^{(k)}}{(k+1)!} + \cdots + g_{1k}^{\dagger}(n)\frac{g_{2n}^{(k)}}{n!} + \dots\right)\lambda^{k-1} \\
&= \alpha_{1_n} + \alpha_{2_n}\lambda + \cdots + \alpha_{k_n}\lambda^{k-1} \\
&= \sum_{j=1}^{k} \alpha_{j_n}\lambda^{j-1},
\end{aligned}$$

where

$$\alpha_{j_n} = \frac{g_{n+k}^{(k)}}{k!} + \frac{g_{n+j-1}^{(k)}}{(j-1)!} + \sum_{i-k+1}^{\infty} g_{1,j}^{\dagger}(i)\frac{g_{n+i}^{(k)}}{i!}$$

for $j = 1, 2, \dots, k$, the proof is completed. □

From theorem 2.3 and theorem 2.6, we have

$$\begin{aligned}
G_n(k,x) &= \sum_{i=0}^{\infty} \frac{g_{n+i}^{(k)}}{i!}x^i \\
&= g_{1,1}^{\dagger}(n-1)G_1(k,x) + \cdots + g_{1,k}^{\dagger}(n-1)G_k(k,x) \\
&= \sum_{j=1}^{k} \alpha_{j_n}x^{j-1},
\end{aligned}$$

where

$$\alpha_{j_n} = \frac{g_{n+k}^{(k)}}{k!} + \frac{g_{n+j-1}^{(k)}}{(j-1)!} + \sum_{i=k+1}^{\infty} g_{1,j}^{\dagger}(i)\frac{g_{n+i}^{(k)}}{i!}$$

for $j = 1, 2, \ldots, k$.

3. SEQUENCES OF GENERALIZED FIBONACCI FUNCTIONS

Matrix methods are a major tool in solving certain problems stemming from linear recurrence relations. In this section, the procedure will be illustrated by means of a sequence, and an interesting example will be given.

To begin with, we introduce the concept of the *resultant* of given polynomials [3]. Let $f(x) = \sum_{i=0}^{n} a_i x^{n-i}$ and $g(x) = \sum_{i=0}^{m} b_i x^{m-i}$ be polynomials, where $a_0 \neq 0$ and $b_0 \neq 0$. The presence of a common divisor for $f(x)$ and $g(x)$ is equivalent to the fact that there exists polynomials $p(x)$ and $q(x)$ such that $f(x)q(x) = g(x)p(x)$ where deg $p(x) \leq n-1$ and deg $q(x) \leq m-1$. Let $q(x) = u_0 x^{m-1} + \cdots + u_{m-1}$ and $p(x) = v_0 x^{n-1} + \cdots + v_{n-1}$. The equality $f(x)q(x) = g(x)p(x)$ can be expressed in the form of a system of equations

$$\begin{aligned} a_0 u_0 &= b_0 v_0 \\ a_1 u_0 + a_0 u_1 &= b_1 v_0 + b_0 v_1 \\ a_2 u_0 + a_1 u_1 + a_0 u_2 &= b_2 v_0 + b_1 v_1 + b_0 v_2 \\ &\vdots \end{aligned}$$

The polynomials $f(x)$ and $g(x)$ have a common root if and only if this system of equations has a nonzero solution $(u_0, u_1, \ldots, v_0, v_1, \ldots)$. If, for example, $m = 3$ and $n = 2$, then the determinant of this system is of the form

$$\begin{vmatrix} a_0 & 0 & 0 & -b_0 & 0 \\ a_1 & a_0 & 0 & -b_1 & -b_0 \\ a_2 & a_1 & a_0 & -b_2 & -b_1 \\ 0 & a_2 & a_1 & -b_3 & -b_2 \\ 0 & 0 & a_2 & 0 & -b_3 \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & a_2 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix} = |S(f(x), g(x))|.$$

The matrix $S(f(x), g(x))$ is called the *Sylvester matrix* of polynomials $f(x)$ and $g(x)$. The determinant of $S(f(x), g(x))$ is called the *resultant* of $f(x)$ and $g(x)$ and is denoted by $R(f(x), g(x))$. It is clear that $R(f(x), g(x)) = 0$ if and only if the polynomials $f(x)$ and $g(x)$ have a common divisor, and hence, an equation $f(x) = 0$ has multiple roots if and only if $R(f(x), f'(x)) = 0$.

Now, we define a sequence. For fixed k, $k \geq 2$, and a complex number a, a sequence of k-Fibonacci functions, $\{G_n(k, a)\}$, is defined recursively as follows:

$$G_0(k, a) = s_0, \; G_1(k, a) = s_1, \; \ldots, \; G_{k-1}(k, a) = s_{k-1}, \tag{6}$$

$$G_n(k,a) = p_1G_{n-1}(k,a) + p_2G_{n-2}(k,a) + \cdots + p_kG_{n-k}(k,a), \quad n \geq k, \tag{7}$$

where $s_0,\ s_1,\ \ldots,\ s_{k-1},\ p_1,\ p_2,\ \ldots,\ p_k$ are complex numbers.

Our natural question now becomes, for $k \geq 2$, what is an explicit expression for $G_n(k,a)$ is terms of $s_0,\ s_1,\ \ldots,\ s_{k-1},\ p_1,\ \ldots,\ p_k$? If $s_0 = \cdots = s_{k-2} = 0,\ s_{k-1} = s_k = 1,\ p_1 = \cdots = p_k = 1$ and $a = 0$, then by theorem 2.2 we have $G_n(k,0) = g_n$. In [8], Rosenbaum gave the explicit expression for $k = 2$.

In this section, we give an explicit expression for $G_n(k,a) = p_1G_{n-1}(k,a)+p_2G_{n-2}(k,a)+\cdots + p_kG_{n-k}(k,a),\ n \geq k$ in terms of initial conditions $G_0(k,a) = s_0,\ G_1(k,a) = s_1,\ \ldots,\ G_{k-1}(k,a) = s_{k-1},\ k \geq 2$.

Let $\tilde{\mathbf{G}}_n(k) = (G_n(k,a),\ \ldots,\ G_{n-k+1}(k,a))^T$ for $k \geq 2$. The fundamental recurrence relation (7) can be defined by the vector recurrence relation $\tilde{\mathbf{G}}_n(k) = \tilde{Q}_k\tilde{\mathbf{G}}_{n-1}(k)$, where

$$\tilde{Q}_k = \begin{bmatrix} \mathbf{p} & p_k \\ I_{k-1} & \mathbf{0} \end{bmatrix} \text{ and } \mathbf{p} = [p_1, p_2, \ldots, p_{k-1}].$$

Let $\mathbf{s} = (s_{k-1}, \ldots, s_0)^T$. Then, we have, for $n \geq 0$, $\tilde{\mathbf{G}}_{n+k-1}(k) = \tilde{Q}_k^n\mathbf{s}$, and the characteristic equation of $\tilde{Q}_k$ is

$$f(\lambda) = \lambda^k - p_1\lambda^{k-1} - \cdots - p_{k-1}\lambda - p_k = 0.$$

If $R(f(\lambda), f'(\lambda)) \neq 0$, then the equation $f(\lambda) = 0$ has distinct k roots.

Theorem 3.1: Let $f(\lambda)$ be the characteristic equation of the matrix $\tilde{Q}_k$. If $R(f(\lambda), f'(\lambda)) \neq 0$, then $G_n(k,a) = p_1G_{n-1}(k,a) + p_2G_{n-2}(k,a) + \cdots + p_kG_{n-k}(k,a)$ has an explicit expression in terms of $s_0, \ldots, s_{k-1}$.

Proof: If $R(f(\lambda), f'(\lambda)) \neq 0$, then the characteristic equation of $\tilde{Q}_k$ has k distinct roots, say $\lambda_1, \lambda_2, \ldots, \lambda_k$. Since the matrix $\tilde{Q}_k$ is diagonalizable, there exists a matrix Λ such that $\Lambda^{-1}\tilde{Q}_k\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Then $\tilde{\mathbf{G}}_{n+k-1}(k) = \Lambda\text{diag}(\lambda_1^n, \lambda_2^n, \ldots, \lambda_k^n)\Lambda^{-1}\mathbf{s}$, and hence we have

$$G_n(k,a) = d_1\lambda_1^n + d_2\lambda_2^n + \cdots + d_k\lambda_k^n = \sum_{i=1}^{k} d_i\lambda_i^n,$$

where $d_1,\ d_2,\ \ldots,\ d_k$ are complex numbers independent of n. We can determine the values of $d_1,\ d_2, \ldots,\ d_k$ by Cramer's rule. That is, by setting $n = 0,\ 1,\ \ldots,\ k-1$, we have

$$\begin{aligned} G_0(k,a) &= d_1 + d_2 + \cdots + d_k, \\ G_1(k,a) &= d_1\lambda_1 + d_2\lambda_2 + \cdots + d_k\lambda_k, \\ &\vdots \\ G_{k-1}(k,a) &= d_1\lambda_1^{k-1} + d_2\lambda_2^{k-1} + \cdots + d_k\lambda_k^{k-1}, \end{aligned}$$

and hence

$$V\mathbf{d} = \mathbf{s}, \quad \mathbf{d} = (d_1, d_2, \ldots, d_k)^T. \tag{8}$$

Therefore, we now have the desired result from (8). □

Recall that

$$\tilde{Q}_k = \begin{bmatrix} \mathbf{p} & p_k \\ I_{k-1} & \mathbf{0} \end{bmatrix},$$

where $[\mathbf{p} = p_1, p_2, \ldots, p_{k-1}]$. Then, in [1], we have the following theorem.

Theorem 3.2 [1]: The (i,j) entry $q_{ij}^{(n)}(p_1, p_2, \ldots, p_k)$ in $\tilde{Q}_k^n$ is given by the following formula:

$$q_{ij}^{(n)}(p_1, p_2, \ldots, p_k) = \sum_{(m_1, \ldots, m_k)} \frac{m_j + m_{j+1} + \cdots + m_k}{m_1 + \cdots + m_k} \times \binom{m_1 + \cdots + m_k}{m_1, m_2, \ldots, m_k} p_1^{m_1} \ldots p_k^{m_k}, \tag{9}$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + km_k = n - i + j$, and the coefficient in (9) is defined to be 1 if $n = i - j$.

Applying the $\tilde{\mathbf{G}}_{n+k-1}(k) = \tilde{Q}_k^n \mathbf{s}$ to the above theorem, we have

$$\begin{aligned} G_n(k,a) &= q_{k1}^{(n)}(p_1, \ldots, p_k) s_{k-1} + q_{k2}^{(n)}(p_1, \ldots, p_k) s_{k-2} + \\ &\quad \cdots + q_{kk}^{(n)}(p_1, \ldots, p_k) s_0 \\ &= \sum_{j=1}^{k} q_{kj}^{(n)}(p_1, \ldots, p_k) s_{k-j}. \end{aligned} \tag{10}$$

From (9), we have

$$q_{kj}^{(n)}(p_1, \ldots, p_k) = \sum_{(m_1, \ldots, m_k)} \frac{m_j + m_{j+1} + \cdots + m_k}{m_1 + \cdots + m_k} \times \binom{m_1 + \cdots + m_k}{m_1, m_2, \ldots, m_k} p_1^{m_1} \ldots p_k^{m_k},$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + km_k = n - k + j$, and the coefficient in (10) is defined to be 1 if $n = k - j$.

Hence, from theorem 3.1 and (10),

$$G_n(k,a) = \sum_{j=1}^{k} q_{kj}^{(n)}(p_1,\dots,p_k)s_{k-j}$$

$$= \sum_{i=1}^{k} d_i\lambda_i^n.$$

Example: In (6) and (7), if we take $a = 0$, $s_0 = s_1 = \cdots = s_{k-3} = 0$, $s_{k-2} = s_{k-1} = 1$ and $p_1 = \cdots = p_k = 1$, then

$$G_0(k,0) = \cdots = G_{k-3}(k,0) = 0,\ G_{k-2}(k,0) = G_{k-1}(k,0) = 1,$$

and for $n \geq k \geq 2$,

$$\begin{aligned} G_n(k,0) &= G_{n-1}(k,0) + G_{n-2}(k,0) + \cdots + G_{n-k}(k,0) \\ &= g_n = g_{n-1} + g_{n-2} + \cdots + g_{n-k}. \end{aligned}$$

Let $\tilde{\mathbf{g}}_n^{(k)} = (g_n^{(k)},\dots,g_{n-k+1}^{(k)})^T$. For any $k \geq 2$, the fundamental recurrence relation $g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}$ can be defined by the vector recurrence relation $\tilde{\mathbf{g}}_n^{(k)} = \tilde{Q}_k\tilde{\mathbf{g}}_{n-1}^{(k)}$. Then, we have $\tilde{\mathbf{g}}_n^{(k)} = \tilde{Q}_k^n\tilde{\mathbf{g}}_0^{(k)} = \tilde{Q}_k^n(1,1,0,\dots,0)^T$. Since $\tilde{Q}_k$ has k distinct eigenvalues (see [5]),

$$g_n^{(k)} = d_1\lambda_1^n + \cdots + d_k\lambda_k^n.$$

Hence, we can determine $d_1, d_2, \dots, d_k$ from (8).

For example, if $k = 3$, then the characteristic equation of $\tilde{Q}_3$ is $f(\lambda) = \lambda^3 - \lambda^2 - \lambda - 1 = 0$, and hence

$$R(f(\lambda), f'(\lambda)) = \begin{vmatrix} 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -1 \\ 3 & -2 & -1 & 0 & 0 \\ 0 & 3 & -2 & -1 & 0 \\ 0 & 0 & 3 & -2 & -1 \end{vmatrix} = 44 \neq 0.$$

Thus $f(\lambda) = 0$ has 3 distinct roots. Suppose α, β and γ are the distinct roots of $f(\lambda) = 0$. Then we have

$$\alpha = \frac{1}{3}(u+v) + \frac{1}{3},$$

$$\beta = -\frac{1}{6}(u+v) + \frac{i\sqrt{3}}{6}(u-v) + \frac{1}{3},$$

$$\gamma = -\frac{1}{6}(u+v) - \frac{i\sqrt{3}}{6}(u-v) + \frac{1}{3},$$

where

$$i = \sqrt{-1}, \quad u = \sqrt[3]{19 + 3\sqrt{33}} \text{ and } \quad v = \sqrt[3]{19 - 3\sqrt{33}}.$$

So, we have

$$g_n^{(3)} = d_1\alpha^n + d_2\beta^n + d_3\gamma^n, \tag{11}$$

and hence

$$\begin{bmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Set

$$\delta = \det \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{bmatrix}, \ \delta_\alpha = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & \beta & \gamma \\ 1 & \beta^2 & \gamma^2 \end{bmatrix}, \ \delta_\beta = \det \begin{bmatrix} 1 & 0 & 1 \\ \alpha & 1 & \gamma \\ \alpha^2 & 1 & \gamma^2 \end{bmatrix},$$

and

$$\delta_\lambda = \det \begin{bmatrix} 1 & 1 & 0 \\ \alpha & \beta & 1 \\ \alpha^2 & \beta^2 & 1 \end{bmatrix}.$$

Then we have

$$d_1 = \frac{\delta_\alpha}{\delta}, \ d_2 = \frac{\delta_\beta}{\delta}, \ \text{ and } d_3 = \frac{\delta_\gamma}{\delta}.$$

As we know, the complex numbers d_1, d_2, and d_3 are independent of n.

We can also find an expression for $g_n^{(3)}$ in [6] follows:

$$g_n^{(3)} = \frac{\left(g_{n-1}^{(3)} + g_{n-2}^{(3)}\right)(\beta - \gamma) - (\beta^n - \alpha^n)}{(\alpha - 1)(\beta - \gamma)}. \tag{12}$$

So, by (11) and (12),

$$\frac{\delta_\alpha\alpha^n + \delta_\beta\beta^n + \delta_\gamma\gamma^n}{\delta} = \frac{\left(g_{n-1}^{(3)} + g_{n-2}^{(3)}\right)(\beta - \gamma) - (\beta^n - \alpha^n)}{(\alpha - 1)(\beta - \gamma)}.$$

Similarly, if $k = 2$, then

$$g_n^{(2)} = F_n = \frac{1}{\lambda_1 - \lambda_2}(\lambda_1^n - \lambda_2^n), \tag{13}$$

where λ_1 and λ_2 are the eigenvalues of Q_2. Actually

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

In this case,

$$d_1 = \frac{1}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}}, \quad d_2 = \frac{1}{\lambda_2 - \lambda_1} = -\frac{1}{\sqrt{5}}$$

and (13) is Binet's formula for the nth Fibonacci number F_n.

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