# A NON-INTEGER PROPERTY OF ELEMENTARY SYMMETRIC FUNCTIONS IN RECIPROCALS OF GENERALISED FIBONACCI NUMBERS


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## 1. INTRODUCTION AND MAIN RESULT

A well-known but classicial result concerning the harmonic series is that the sequence of partial sums $\sum_{r=1}^{n} \frac{1}{r}$ can never be an integer for $n > 1$. More generally, Nagell [3] showed that $\sum_{r=1}^{n} \frac{1}{m+rd}$ cannot be an integer for any positive integers $m$, $n$ and $d$. As an extension of these results the author, in a recent paper [4], constructed further examples of positive rational termed series having non-integer partial sums. These partial sums were of the form $\sum_{r=1}^{n} \frac{1}{U_r}$, where $\{U_n\}$ are the sequence of generalised Fibonacci numbers generated, for $n \geq 2$, via the recurrence relation

$$U_n = PU_{n-1} - QU_{n-2}, \tag{1}$$

with $U_0 = 0$, $U_1 = 1$ and $(P, Q)$ a relatively prime pair of integers satisfying $|P| > Q > 0$ or $P \neq 0$, $Q < 0$. (Note when $(P, Q) = (2, 1)$ one has $U_n = n$). By viewing these partial sums as the symmetric function formed from summing the products of the terms $\frac{1}{U_1}, \frac{1}{U_2}, \dots \frac{1}{U_n}$ taken one at a time, one may naturally ask whether all other symmetric fucntions in the reciprocals of such generalised Fibonacci numbers can be non-integer. In this paper we will show that for sequences $\{U_n\}$ generated via (1), with $P \geq 2$ and $Q < 0$, there can in fact be at most finitely many $n$ such that one or more of the elementary symmetric functions in $\frac{1}{U_1}, \frac{1}{U_2}, \dots \frac{1}{U_n}$ is an integer. To establish this result we will require two preliminary Lemmas, the first of which is a refinement of Bertrand's postulate due to Ingham [2].

**Lemma 1.1**: *For any real number $x > 1$ there always exists a prime in the interval $(x, x+x^{\frac{5}{8}})$.*

The second lemma is a standard result of generalised Fibonacci sequences, a proof of which can be found in [1].

**Lemma 1.2**: *For any sequence $\{U_n\}$ generated with respect to a relatively prime pair of integers $(P, Q)$ via (1) then $(U_m, U_n) = U_{(m,n)}$.*

We now can prove the following theorem:

**Theorem 1.1**: *Suppose the sequence $\{U_n\}$ is generated via (1) with respect to the relatively prime pair $(P, Q)$ such that $P \geq 2$ and $Q < 0$. Denote the $k^{th}$ elementary symmetric function in $\frac{1}{U_1}, \frac{1}{U_2}, \dots, \frac{1}{U_n}$ by $\phi(n, k)$, then for this family of functions there exists a uniform lower bound $N$ on $n$, such that $\phi(n, k)$ is non-integer for $n \geq N$ and $1 \leq k \leq n$.*

**Proof**: To establish the non-integer status of $\phi(n, k)$ it will suffice to consider the two separate cases of $k > 3\log n$ and $k < 3\log n$, noting here that it is sufficient to take only strict inequalities as $\log n$ can never be an integer for integer $n > 1$. In both cases we will demonstrate the existence of the lower bounds given by $N_1 = \min\{s \in \mathbb{N} : \log n \geq \frac{e}{3-e}$ for $n \geq s\} = \lceil e^{\frac{e}{3-e}} \rceil$ and $N_2 = \min\{s \in \mathbb{N} : \frac{9(\log n)^2}{n} + \frac{3\log n}{n} < \frac{1}{2}, \frac{n^3}{(3\log n+1)^{11}} > 2^8(1+\log 3)^5$ for all $n \geq s\}$ respectively on $n$, for which $\phi(n,k)$ is non-integer. As $N_1$ and $N_2$ are constructed independently of $k$, one can then set $N = \max\{N_1, N_2\}$ from which it is immeidate that $\phi(n,k)$ must be non-integer for all $n \geq N$ and $1 \leq k \leq n$. Furthermore, as $N_1$ and $N_2$ are not dependent on the specific choice of the sequence $\{U_n\}$, one sees that the lower bound $N$ must hold uniformly over the family of generalised Fibonacci sequences as specified in the theorem statement. We now proceed with the following two cases.

**Case 1**: $k > 3\log n$

First note for the prescribed values of $(P,Q)$ it can be shown, via an easy induction on $n$, that $U_n \geq n$. Now, as $\phi(n,k)$ is formed from summing the terms $\frac{1}{U_1}, \frac{1}{U_2}, \ldots, \frac{1}{U_n}$ taken $k$ at a time, we observe that $\phi(n,k)$ must occur $k!$ times in the multinomial expansion $\left(\frac{1}{U_1} + \frac{1}{U_2} + \cdots + \frac{1}{U_n}\right)^k$. Hence, using the usual comparison of $\log n$ with the terms of the harmonic series, we obtain that

$$\phi(n,k) < \frac{1}{k!}\left(\frac{1}{U_1} + \frac{1}{U_2} + \cdots + \frac{1}{U_n}\right)^k < \frac{1}{k!}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)^k$$
$$< \frac{1}{k!}(1+\log n)^k. \tag{2}$$

Now by definition of $N_1$ if $n \geq N_1$ then $\log n > \frac{e}{3-e}$ and so $k > \frac{3e}{3-e}$. Consequently

$$\frac{1}{k!}(1+\log n)^k < \frac{1}{k!}\left(1+\frac{k}{3}\right)^k = \frac{k^k}{k!}\left(\frac{1}{k}+\frac{1}{3}\right)^k < \left(\frac{e}{k}+\frac{e}{3}\right)^k < 1,$$

noting here that the second last inequality follows from the fact that $\frac{k^k}{k!} < e^k$. Hence, we deduce from the previous inequality and (2) that $0 < \phi(n,k) < 1$ for any $n \geq N_1$ as required.

**Case 2**: $k < 3\log n$

In this case it first will be necessary to show that for $n \geq N_2$

$$\left(\frac{n}{k(k+1)} - 1\right)^8 > \left(\frac{n}{k}+1\right)^5. \tag{3}$$

Upon factoring out $\frac{n}{k}$ and $\frac{n}{k(k+1)}$ from the right and left hand side respectively of the conjectured inequality in (3) one finds that

$$\frac{n^3}{k^3(k+1)^8}\left(1 - \frac{k(k+1)}{n}\right)^8 > \left(1+\frac{k}{n}\right)^5. \tag{4}$$

Now, as $k < 3\log n$ and so $\frac{k}{n} < \frac{3\log n}{n} \to 0$ monotonically for $n > e$, it is clear the term $\left(1+\frac{k}{n}\right)^5$ can be bounded above by $(1+\log 3)^5$ for $n \geq 3$ say. Similarly, as $\frac{k(k+1)}{n} < \frac{9(\log n)^2}{n} + \frac{3\log n}{n} \to 0$ and $\frac{n^3}{k^3(k+1)^8} > \frac{n^3}{(3\log n+1)^{11}} \to \infty$ as $n \to \infty$, one can choose $n$ sufficiently large but finite and independent of $k$, such that $\frac{k(k+1)}{n} < \frac{1}{2}$ and $\frac{n^3}{k^3(k+1)^8} > 2^8(1+\log 3)^5$. Consequently by definition of $N_2$ one has for $n \geq N_2$

$$\frac{n^3}{k^3(k+1)^8}\left(1-\frac{k(k+1)}{n}\right)^8 > (1+\log 3)^5$$

and so one concludes that (3) must hold for all $n \geq N_2$. Now raising both sides of (3) to the power $\frac{1}{8}$ one finds upon rearrangement that

$$\frac{n}{k} > \left(1+\frac{n}{k+1}\right) + \left(1+\frac{n}{k+1}\right)^{\frac{5}{8}}.$$

Hence for $n \geq N_2$ there must exist, by Lemma 1.1, a prime $p$ in the open interval $\left(1+\frac{n}{k+1}, \frac{n}{k}\right)$. By construction $p$ must be such that $1 < mp < n$ for $m = 1, 2, \ldots, k$ but $(k+1)p > n$. Considering again $\phi(n,k)$ as a sum of the product of the terms $\frac{1}{U_1}, \frac{1}{U_2}, \ldots, \frac{1}{U_n}$ taken $k$ at a time we can write

$$\phi(n,k) = \sum_{i=1}^{\binom{n}{k}} \frac{1}{c_i} = \frac{b_1 + b_2 + \cdots + b_{\binom{n}{k}}}{U_1U_2\ldots U_n} = \frac{B}{C},$$

where $c_i$ is one of the possible $\binom{n}{k}$ products of the terms $U_1, U_2, \ldots, U_n$ taken $k$ at a time and

$$b_i = \frac{U_1U_2\ldots U_n}{c_i}.$$

By the above $U_pU_{2p}\ldots U_{kp} = c_s$, for some $s \in \{1, 2, \ldots, \binom{n}{k}\}$, and as $(k+1)p > n$, no other of the remaining $\binom{n}{k} - 1$ products $c_i$ can contain generalised Fibonacci numbers in which all of the corresponding $k$ subscripts are a multiple of $p$. Consequently, by construction each $b_i$, with

$i \neq s$, must contain at least one of the terms in the set $A = \{U_p, U_{2p}, \ldots, U_{kp}\}$ while $b_s$ will contain none of the terms in $A$. Now by Lemma 1.2 as $p$ is prime $(U_p, U_{mp}) = U_{(p,mp)} = U_p$, for each $m = 1, 2, \ldots, k$, and so $U_p | b_i$ for every $i \neq s$. Also for $(r,p) = 1$ one has $(U_p, U_r) = U_1 = 1$ but as $b_s$ contains only those terms $U_r$ for which $(r,p) = 1$, we conclude that $U_p$ must be relatively prime to $b_s$, and so $U_p \not| b_s$, which in turn implies that $U_p \not| B$. Thus $\phi(n,k) = \frac{B}{C}$ where $U_p | C$ but $U_p \not| B$, that is $\phi(n,k)$ cannot be an integer for any $n \geq N_2$ as required. □

**Remark 1.1**: *It is clear that the above argument could easily be applied to higher order recurrences $\{U_n\}$ with $U_n \geq n$ if an analogous result in Lemma 1.2 could be found.*

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## REFERENCES

[1] R. D. Carmichael. "On the Numerical Factors of the Arithmetic Form $\alpha^n \pm \beta^n$." *Annals of Mathematics* **15** (1931): 30-70.

[2] A. E. Ingham. "On the Difference Between Consecutive Primes." *Quart. J. Math Oxford Ser.* **8** (1937): 255-266.

[3] T. Nagel. "Eine Eigenschaft gewissen Summen." *Skr. Norske Vid. Akad. Kristiania* **13** (1923): 10-15.

[4] M.A. Nyblom. "On Divergent Series with Non-Integer Partial Sums." *Punjab Univ. J. Math. (Lahore)* **31** (1998): 55-67.

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